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Social networks, confirmation bias and shock elections

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In recent years online social networks have become increasingly prominent in political campaigns and, concurrently, several countries have experienced shock election outcomes. This paper proposes a model that links these two phenomena. In our set-up, the process of learning from others on a network is influenced by confirmation bias, i.e. the tendency to ignore contrary evidence and interpret it as consistent with one's own belief. When agents pay enough attention to themselves, confirmation bias leads to slower learning in any symmetric network, and it increases polarization in society. We identify a subset of agents that become more/less influential with confirmation bias. The socially optimal network structure depends critically on the information available to the social planner. When she cannot observe agents' beliefs, the optimal network is symmetric, vertex-transitive and has no self-loops. We explore the implications of these results for electoral outcomes and media markets. Confirmation bias increases the likelihood of shock elections, and it pushes fringe media to take a more extreme ideology.

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The strongest bias in American politics is not a liberal bias or a conservative bias; it is a confirmation bias, or the urge to believe only things that confirm what you already believe to be true. Not only do we tend to seek out and remember information that reaffirms what we already believe, but there is also a “backfire effect” which sees people doubling down on their beliefs after being presented with evidence that contradicts them.

“Your facts or mine?” by E. Roller, *NYT*, Oct 25th, 2016.

Social networks are increasingly becoming the primary channel for people to acquire information and form opinions. In an experiment involving 60m Facebook users prior to the 2010 US elections, Bond et al. [2012] showed they could generate 340,000 additional votes using a social message that informed a user about friends that had voted, compared to an informational message without social network information. Unlike traditional media or gatherings in the local church, club or pub, online social networks make it very easy for a user to “unfollow” someone who does not share their opinion. In this way, they exacerbate the role of confirmation bias – people’s tendency to ignore information contrary to their view and reinterpret it as agreeing with their own (Pariser [2011]). A natural question is whether the growing importance of online social networks in opinion formation, and the corresponding heightened role of confirmation bias, affects the democratic process and whether it is a driver of shock elections results such as Trump’s win or Brexit.

The aim of this paper is to examine how confirmation bias affects the process of learning from others, and its consequences for elections and the media. We are interested in learning at the societal level, so individuals are embedded in a large network. They are endowed with an initial belief at time 0 and learn according to the well-known DeGroot [1974] behavioral rule: they update their beliefs by taking weighted averages of their neighbours’ beliefs. Experimental evidence shows that this DeGroot learning rule is a good predictor of how people learn from others, and it is particularly suitable to model learning in a large network where it is unrealistic to assume individuals are Bayesian updating and process information conditioning on the network structure.¹

In the spirit of Rabin and Schrag [1999], confirmation bias in the model implies that when an individual learns that someone has beliefs too different from their own, they ignore them thereafter and give more weight to their own belief instead. Specifically, after the assignment of initial beliefs each individual cuts connections with others who have beliefs further away from their own than a threshold. The individual reassigns the weight of these severed connections to themselves, and they never reinstate these links after they have been cut. Mathematically, understanding the effect of confirmation bias in the model reduces to a comparison of the learning processes on the original network and the new sparser network.

The first result is that, when agents pay enough attention to themselves, confirmation bias slows down learning in *any* symmetric network. In the proof we apply the notion of Dirichlet

¹See, e.g., Corazzini et al. [2012] and Chandrasekhar et al. [2020]. Section 17.2.5 in Choi et al. [2016] reviews experimental evidence on social learning.

energy to relate the confirmation bias parameter to the whole spectrum of eigenvalues of the network matrix, which governs the rate of convergence to a consensus. Agents need to pay enough attention to themselves so that their beliefs do not oscillate from period to period. Mathematically, this restriction ensures that the eigenvalues are positive, which guarantees that changes in size correspond to changes in magnitude. Using counterexamples we show that the result is not just a consequence of the network being sparser, but it critically relies on the key feature of confirmation bias that the individual puts more weight on their *own* belief: if the weight of the severed links is partly reassigned to other surviving links then learning may be faster or slower. While this result requires the network to be symmetric, we show using simulations that it largely holds in asymmetric networks too.

Confirmation bias leads to a redistribution of influence. The intuitive result that individuals who cut links increase their influence does not always hold. We can, however, show that there are individuals that we dub influencers (listeners) whose influence increases (decreases) in the presence of confirmation bias. A further consequence of confirmation bias is that society becomes more polarized at each point in time.

A natural objective of a social planner is to maximize the chance that society converges to the truth. Confirmation bias works against this goal by redistributing influence across individuals and, potentially, breaking the network into separate components – preventing the aggregation of initial signals. Assuming that a social planner does not observe the distribution of initial signals or the level of confirmation bias, we characterize the set of networks that maximize the probability that a society converges to the truth. Given a fixed budget of links to allocate, optimal networks are symmetric, have no self-links, and their unweighted equivalent is vertex-transitive – a subset of regular networks such that every node is structurally equivalent to every other in the network.

In the second part of the paper, we examine the consequences of social learning affected by confirmation bias for elections and media markets. In the first application, we embed our social learning framework into a two-candidate voting model in which sincere voting is a weakly dominant strategy so whenever there is an election individuals vote for the candidate closer to their current belief. We restrict the distribution of beliefs to focus on the interesting case when a society would vote for the same candidate before learning takes place and at the end of the learning process. We define a society as having *shock elections* if the other candidate wins at any point in time during the learning process. Using a mean-field assumption, we prove that a society never has shock elections without confirmation bias, but it can do if confirmation bias is high enough to remove some connections. Simulations show that this result holds even without the mean-field assumption.

Finally, we embed our social learning framework in a Hotelling-style model of a media market. Media players choose their editorial line, or ideology, and only care about maximizing their audience. Individuals follow one and only one media organization due to their limited attention budgets. We focus on “fringe” media organizations that adopt an extreme editorial line, and prove

the editorial line of the fringe media organization becomes more extreme as the strength of confirmation bias increases.

Literature review. This paper sits at the intersection of literatures in behavioral economics, social learning and political economy. We review each in turn, highlighting papers that are particularly relevant to this work.

Psychology. The study of confirmation bias has a long history in psychology; a comprehensive review by Nickerson [1998] shows its relevance to a large range of issues, including judicial outcomes (Kuhn et al. [1994]), policymaking (Tuchman [2011]), and medical decisions (Elstein and Bordage [1979]).² A difficulty posed by the vastness of this literature is, quoting Nickerson, that “confirmation bias has been used in the psychological literature to refer to a variety of phenomena” (p. 175). Nickerson [1998]’s working definition is “unwitting selectivity in the acquisition and use of evidence” (p. 175) and he puts special emphasis on the non-deliberate nature of the bias which emerges as a heuristic to quickly process information. The core idea present throughout the psychology literature is that people are biased against information which conflicts with their own beliefs.

Behavioral economics. In the economics literature, Rabin and Schrag [1999] formulate a model of how confirmation bias affects individual decision-making. In their set-up, there are two states of the world. Each time a new signal arrives, the agent performs Bayesian updating, with the twist that when a signal runs counter to the agent’s current hypothesis then there is a probability q that the agent misinterprets it as actually confirming her hypothesis; i.e. q is the strength of confirmation bias. The main result of their paper is that confirmation bias leads to overconfidence. Epstein [2006] presents a model of Non-Bayesian updating, where the agent is ‘tempted’ to change their belief after receiving a signal.³ This model is able to nest a version of confirmation bias by choosing an appropriate specification of how to agent is tempted to update their belief.

The main contribution of our paper to behavioral economics is to analyze the effects of confirmation bias in processing information in a context with multiple agents who learn from each other through their social connections. This is arguably becoming more relevant nowadays, as individuals increasingly learn by sharing information on social media rather than individually processing information from a media source. A crucial step is to model confirmation bias as reducing the range of opinions an individual is willing to listen to. This translates to ignoring information in a similar way to the single agent in Rabin and Schrag [1999]’s framework. In our basic model, an

²Nickerson [1998]’s opening paragraphs states: “If one were to attempt to identify a single problematic aspect of human reasoning that deserves attention above all others, the confirmation bias would have to be among the candidates for consideration. Many have written about this bias, and it appears to be sufficiently strong and pervasive that one is led to wonder whether the bias, by itself, might account for a significant fraction of the disputes, altercations, and misunderstandings that occur among individuals, groups, and nations.” The careful reader will note that the pervasiveness of confirmation bias may extend to the quote itself.

³It is similar to Gul and Pesendorfer [2001]’s model except that it is *beliefs*, rather than utilities, that change.

agent always ignores information too far away from their view rather than only some of the time, as in Rabin and Schrag [1999]. Relaxing this assumption does not affect the main results.⁴

Social learning. Research on social learning in economics began with the seminal papers by Banerjee [1992] and Bikhchandani et al. [1992] in which rational players take actions in succession and each mover can see the actions of their predecessors. A sizeable branch of subsequent work has enriched this basic framework by embedding agents in a network and relaxing the assumption of sequential moves. In this more complex set-up, however, tractability is a challenge and assuming full Bayesian rationality tends to limit the results to showing convergence to consensus in the long-run (see Golub and Sadler [2016] for a comprehensive review). Moreover, the sophistication required by Bayesian reasoning in this set-up is unrealistic in large societies, and Corazzini et al. [2012] show experimentally that even in small groups it is a poor predictor of how individuals learn.

An alternative approach is to assume agents are non-Bayesian and use a behavioral rule to learn from others. DeGroot [1974] proposed the simple rule that agents update their beliefs by taking a weighted average of their neighbours’ opinions, and he shows that the process reaches a consensus under mild regularity conditions. This set-up gained traction in economics with DeMarzo et al. [2003], who obtain novel results on convergence speed and relate each agent’s contribution to the consensus to their respective network position. More recently, Golub and Jackson [2010] give new results on network structures that lead to society correctly aggregating information.

The primary contribution of our paper to the social learning literature is to study how confirmation bias affects the outcomes of the learning process, including convergence to a consensus, speed of learning and the influence of agents. To our knowledge, this is the first paper to examine how a psychological bias affects DeGroot-type learning. Aside from its intrinsic interest and applications, it also provides a check on whether existing DeGroot-type learning results are robust to a bias that is ubiquitous in reality. The objective is similar to Golub and Jackson [2012]: they examine how homophily, a ubiquitous feature of social networks, affects the speed of social learning and consensus.⁵

Voting. In recent years political experts have been surprised by several electoral outcomes, including the election of Donald Trump in the United States, and the outcome of the U.K.’s Brexit referendum. It may not be entirely coincidental that these events occurred alongside a shift in news consumption from traditional media outlets to online social networks (Gottfried and Shearer [2016]). For instance, an extensive study of 10.1 million U.S. Facebook users by Bakshy et al. [2015] shows that people tend to predominantly share news with friends that is in line with the recipient’s ideology, and this filtering by friends is more powerful than Facebook’s algorithmic selection on the news feed to limit exposure to distant viewpoints.

Alongside this is a growing literature on ‘fake news’ and its potential impact on elections.

⁴See Appendix A.1 for further discussion.

⁵Other papers using the DeGroot framework include Acemoglu et al. [2010], Gallo [2014] and Jadbabaie et al. [2012].

Allcott and Gentzkow [2017] provide an overview of some recent work, and also sketch a model of fake news. They suggest that if agents have a preference for confirmatory news reporting, then news reporting can become distorted, possibly reducing the ability of democracies to choose high-quality candidates.

This paper shows how confirmation bias’ impact on the way we learn from others can lead to surprising election outcomes. In particular, confirmation bias prevents us from directly learning from other people whose information conflicts with our own views. This means that our immediate friends are unrepresentative of society as a whole. Therefore, we can be swayed one way in the medium term, even though the weight of information on aggregate points the other. A consequence of this in the short/medium-term is that a society may vote for a candidate that would not be supported by the majority once long-run information aggregation has occurred. Confirmation bias implies that a society can choose policies that, in the long term, it would not want.

1 Model

This section presents the main elements of the model: the network and initial signals, the learning process, and the way we model confirmation bias.

Endowments. Consider a finite set of agents $N = \{1, 2, \dots, n\}$ who communicate through a directed, weighted network $T \in \mathcal{T}$. \mathcal{T} is the set of all networks with n nodes. The entry $T_{ij} \in [0, 1]$ denotes how much weight agent i places on the views of agent j and $\sum_{j \in N} T_{ij} = 1$ for all i ; so T is row-stochastic. The *self-link* T_{ii} is the weight an agent places on her own view. A directed path of length l between i and j is a sequence of links $T_{ik_1}, \dots, T_{k_{l-1}j}$ such that no two nodes on the path are the same. We assume that T is *strongly connected* – there is a directed path from any agent to any other agent – and *aperiodic* – there are no cycles.

We say that agent i listens to j if $T_{ij} > 0$, and i is listened to by j if $T_{ji} > 0$. Denote by $N_i^{out}(T) = \{j \in N | T_{ij} > 0\}$ the out-neighbourhood and by $d_i^{out}(T) = |N_i^{out}(T)|$ the *out-degree* of agent i . Similarly, $N_i^{in}(T) = \{j \in N | T_{ji} > 0\}$ is the in-neighbourhood and $d_i^{in}(T) = |N_i^{in}(T)|$ the *in-degree* of agent i . A network is *symmetric* if $T_{ij} = T_{ji}$ for all $i, j \in N$. If it is symmetric, then $d_i^{in}(T) = d_i^{out}(T)$ for all $i \in N$. It is *regular* if $d_i^{in}(T) = d^{in}(T)$ and $d_i^{out}(T) = d^{out}(T)$ for all $i \in N$.

Each agent is endowed with a signal $\theta_i \in [0, 1]$ about the underlying state of the world, and we make the standard assumption in the literature that agent’s i initial belief x_{i0} at time $t = 0$ is equal to the initial signal received by i .⁶ Notice that the initial belief x_{i0} of agent i is independent of i ’s position in network T .

Learning. In each time period an agent updates her belief by taking a weighted average of her current belief and the beliefs of agents she listens to. Mathematically, agents’ beliefs at time t are

⁶We leave the distribution of signals unspecified because it does not matter for the results in the paper.

equal to $\mathbf{x}_t = T^* \mathbf{x}_{t-1}$. Iterating, we have that $\mathbf{x}_t = (T^*)^t \mathbf{x}_0$ so we can derive agents' beliefs at time t from the initial signals and the network.

Confirmation bias. In the first step of the learning process, agents truthfully share their signals and therefore an agent learns the initial belief of their neighbours. We assume that when an agent with confirmation bias learns that the difference between a neighbour's belief and her own belief exceeds a threshold, she ignores that neighbour and transfers the weight she would have put on that neighbour's belief to her own belief. Moreover, she never listens again to information from that neighbour for the rest of the learning process.

Definition 1. A society on a network T in which agents have confirmation bias q communicates according to a network T^* such that:

$$\begin{aligned} \text{if } |x_{i0} - x_{j0}| > (1 - q) \quad & \text{then} \quad T_{ij}^* = 0 \quad T_{ii}^* = T_{ii} + T_{ij} \\ \text{otherwise} \quad & T_{ij}^* = T_{ij} \quad T_{ii}^* = T_{ii} \end{aligned}$$

Mathematically, a society in which agents have confirmation bias q communicates through a network T^* that has had links cut compared to T , and where the weight of the links that have been cut is redistributed to self-links. Notice that the threshold is defined as $1 - q$ so that a higher q corresponds to increasing confirmation bias. Understanding the impact of confirmation bias in the model means comparing the learning processes on T and T^* . Unless stated otherwise, we assume that T and T^* are both strongly connected. There are some implicit assumptions in modeling confirmation bias in this way which are worth discussing and motivating upfront.

1. Agents *completely* cut links. This is a realistic assumption for online social networks where users have to make a binary decision on whether to follow or unfollow someone. The results are, however, robust to relaxing this assumption. In Appendix A.1 we show that the main result concerning the speed of learning holds if agents affected by confirmation bias weaken rather than cut links, and the weakening can be either by a common factor or proportionally to the difference in beliefs.
2. Agents *redistribute* weights of cut links *to themselves*. This is an important feature of confirmation bias from a large body of experimental evidence (Nickerson [1998]) as well as consistent with the “backfire effect” mentioned in the initial quote. If agents were to redistribute the weight to, say, other neighbours then this would not be confirmation bias and, as we will show, it would lead to different results.
3. Agents *never reinstate* links they have cut. This is consistent with the standard assumption of myopic learning in the DeGroot model, in which the weights an agent gives to different neighbours are fixed at the beginning and never change over time. It also helps in terms of

tractability because time-varying weights would significantly increase the complexity of the Markov process.

2 How confirmation bias affects learning

This section examines how confirmation bias affects social learning. Section 2.1 considers speed of learning, section 2.2 considers agents' influence, and section 2.3 considers polarization of beliefs. Finally, section 2.4 characterizes the optimal networks to minimize the adverse consequences of confirmation bias on the learning process.

2.1 Speed of Learning

The assumptions that T and T^* are strongly connected and T is aperiodic ensure that in the long run the society converges to a consensus where all agents agree with one another. Reaching a consensus may, however, take a long time and the purpose of this section is to characterize how this convergence time varies with confirmation bias.

In the Markov chain literature there are different definitions of convergence or mixing time. In general, mixing time depends on the spectrum of eigenvalues, which is often well-approximated by the second largest eigenvalue (Montenegro et al. [2006]). In this paper we adopt the following definition of convergence time that takes into account the full spectrum of eigenvalues of the matrix T .

Definition 2. The *average convergence time* τ is equal to:

$$\tau = \min \left\{ t > 0 : \frac{1}{n} \sum_i \|T^t(i, \cdot) - s\|_2^2 < \epsilon \right\}$$

where $\|\cdot\|_2$ denotes the ℓ_2 norm.

Average convergence time captures how long it takes for agents, on average, to get within a distance ϵ of their invariant distribution – the proportion of total attention they indirectly pay to all agents in the long run. Notice that the initial assignment of beliefs does not enter explicitly in the definition. In economics terms, what we care about is the speed of convergence of beliefs to a consensus and the initial assignment matters for this – if an agent's initial belief is extreme then it will take more time for her to converge to the consensus. The evolution of beliefs, however, closely tracks, on average, the evolution of our measure, and we validate this in a simulation study in section 5. Moreover, Appendix A.1 shows that our results are robust to adopting the definition of convergence in Golub and Jackson [2010] based on the second eigenvalue and the worst-case scenario in the assignment of initial beliefs.

Armed with this definition, we can prove that confirmation bias always weakly increases convergence time in any symmetric network provided that agents listen mostly to themselves.

Theorem 1. *When $T_{ii} \geq \frac{1}{2}$ for all i , then for any symmetric network T , the average convergence time τ is (weakly) monotonically increasing in the amount of confirmation bias q .*

The proof consists of two steps. First, we show that the Dirichlet energy of T^* is lower than the one of T . Second, we rely on results from the Markov chain literature to relate the Dirichlet energy to the whole spectrum of eigenvalues of T and T^* . The result holds for the large class of symmetric networks, which captures many types of real social networks, such as trust, friendship and family networks. It does, however, exclude some other types of social networks, such as Twitter. The limitation to symmetric networks is necessary for tractability because it implies that the removal of links does not change the influence each agent has on the final consensus outcome. In other words, it simplifies the analysis by disentangling the effect of confirmation bias on convergence time from its effect on the distribution of influence, which may also affect convergence time and is the focus of the next section. The simulations in section 5 show that the result in Theorem 1 largely holds in asymmetric networks as well.

From a technical standpoint, the restriction that $T_{ii} \geq \frac{1}{2}$ for all i ensure the Markov chain has no negative eigenvalues – this is a standard approach in the mathematics literature.⁷ It ensures we avoid situations where the Markov chain is “nearly” periodic. Intuitively, we can think of the periods in the model as corresponding to relatively short periods of chronological time, so it would be unreasonable for agents to have large swings in their belief from one period to the next.

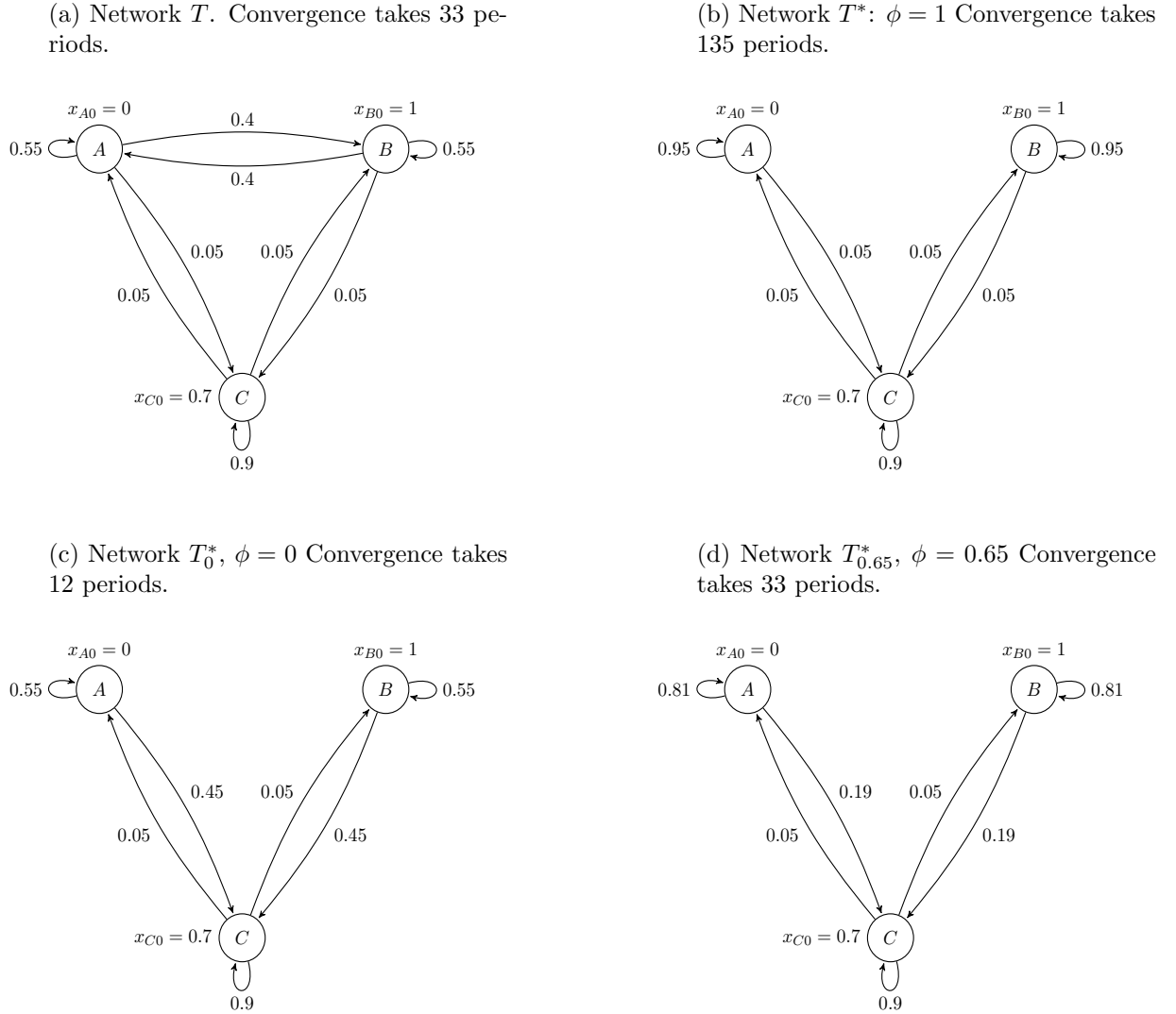
A tempting interpretation of Theorem 1 is that convergence time is longer with confirmation bias simply because T^* is a sparser network than T . This is not correct. Recall that there are two features that define what it means to have confirmation bias. The first one is that someone with confirmation bias ignores information from others whose beliefs are too different – this is the removal of links that makes the network sparser. The second one is what the initial quote dubs the backfire effect – the weight of these links is fully redirected to the self-link. The counterexample in Figure 1 shows that both features of confirmation bias are essential for the result in Theorem 1 to hold.

In particular, consider the following extended model that nests confirmation bias as a special case. As in the current set-up, i removes a link with j if $|x_{i0} - x_{j0}| > 1 - q$. The extension is that i reroutes a fraction $\phi \in [0, 1]$ of the severed link to herself, and spreads out a fraction $1 - \phi$ across her remaining links in proportion to the strength of each link. Clearly, $\phi = 1$ is the special case in which the rerouting of links is determined by confirmation bias. Consider network T and the allocation of initial beliefs in Figure 1a. If $q \in [0, 0.3]$ and $\phi = 1$, so rerouting of links is determined by confirmation bias, then the resulting network is T^* in Figure 1b. As expected from Theorem

⁷See for example Levin et al. [2009], McNew [2011], Basu et al. [2014]. This assumption is sufficient but not necessary. The Markov chain needs only to be positive semi-definite.

1, the convergence time in T^* is 135 periods, which is longer than the 33 periods required in T when there is no confirmation bias. If, instead, $\phi = 0$ then the resulting network T_0^* is displayed in Figure 1c and convergence time is 12 periods. Notice that T_0^* is *sparser* than the initial network T , but convergence in T_0^* is *faster* than in T . It turns out that convergence is faster for most values of ϕ , Figure 1d displays network $T_{0.65}^*$ for the critical value of $\phi = 0.65$ such that the convergence time is the same as in T – for any $\phi < 0.65$ the convergence time is faster than in T even though all these networks are sparser than T .⁸

Figure 1



⁸When considering the average consensus time – which depends on the sum of squared eigenvalues (see Appendix B) and abstracts from the initial beliefs – this counterexample holds for the range $\phi \in [0, 0.27]$.

2.2 Influence and influencers

A single Bayesian agent who aggregates all the information in a society would weight each initial signal equally in their posterior. However, when the society learns through DeGroot social learning, the weight that an agent's initial signal has in the final consensus depends on that agent's network position. Confirmation bias alters the network structure, so it may affect an agent's influence on the consensus. Agents who cut links put more weight on their initial signal and less on others' beliefs. Intuitively, we might expect that this means they have greater influence on the final consensus. This section examines how confirmation bias changes and redistributes influence, and shows that this intuition does not follow through.

An appealing feature of the DeGroot framework is that an agent's influence is equal to their eigenvector centrality, and this notion is captured by the following standard definition.

Definition 3. The influence, s_i , of agent i is the i^{th} entry in the left-hand unit eigenvector associated with the first eigenvalue $\lambda_1 \equiv 1$:

$$s \lambda_1 = s T \text{ where } s = (s_1, s_2, \dots, s_n) \implies s_i = \sum_{j=1}^n T_{ji} s_j$$

Our first result identifies a class of networks in which confirmation bias does not affect agents' influence on the final consensus.

Remark 1. If T is symmetric and T^* is strongly connected, then all agents have equal influence and confirmation bias does not alter any agent's influence.

In a symmetric network all agents have the same influence.⁹ Given that the level of confirmation bias q is the same for all agents, if i cuts the T_{ij} link then j will also cut the T_{ji} link. The resulting network T^* is, as a consequence, also symmetric, and therefore all agents will continue to have the same influence as in T .

When the network is not symmetric, agents' influence varies with their position in the network. Because confirmation bias removes some links from the network, it will also redistribute the influence across agents. The following remark shows that when there is just a single agent who cuts links due to confirmation bias, their influence unambiguously rises.

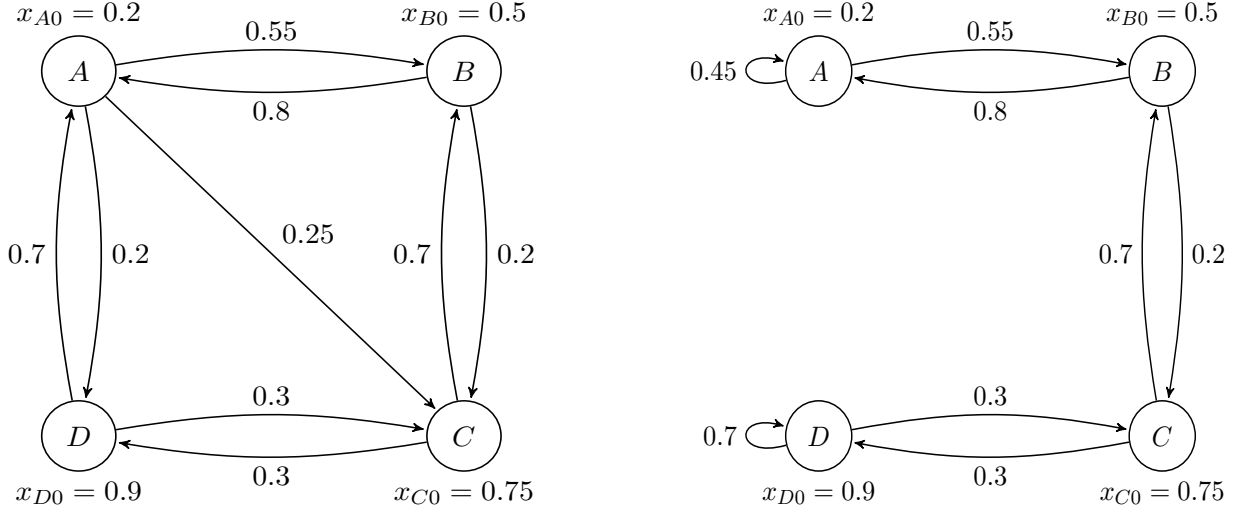
Remark 2. If exactly one agent i cuts one or more links, then their influence, s_i , strictly increases.

An appealing conjecture is that this extends to a situation where many agents cut links. Unfortunately, Figure 2 demonstrates that this is not the case. When the level of confirmation bias is in the range $q \in [0.3, 0.7)$, and initial beliefs are $(0.2, 0.5, 0.75, 0.9)$, the listening structure T on

⁹This is because the first left-hand eigenvector of any symmetric $n \times n$ Markov chain is a vector with all entries $1/n$.

the left changes to T^* on the right because A removes links with C and D , and D removes the link with A . Among these agents who cut links, A 's influence rises in T^* , but D 's influence *decreases* in T^* despite the fact she has removed a link. The reason for this decrease is that in T^* the most influential agent A does not listen to D any more, and therefore D does not directly affect the beliefs of the most influential agent. Conversely, B does not cut any links but her influence *rises* because she is the only agent in T^* that the influential agent A still listens to.

Figure 2: Original network T on the left and resulting network T^* with confirmation bias in the $[0.3, 0.7)$ range.



(a) Network T :
 $s = (0.353, 0.328, 0.192, 0.128)'$

(b) Network T^* :
 $s^* = (0.481, 0.330, 0.094, 0.094)'$

The message from the counterexample is that understanding the change in an agent i 's influence involves keeping track of the change in influence of the agents who listen to i , in addition to whether i cuts any links. Definition 4 captures this notion by identifying an *influencer* as someone who (1) cuts links with other agents, but (2) no other agent cuts a link with her, and (3) continues listening only to agents who satisfy (1) and (2).

Definition 4. An agent i is an *influencer* if $d_{i,\text{out}}(T^*) < d_{i,\text{out}}(T)$ and $d_{i,\text{in}}(T^*) = d_{i,\text{in}}(T)$; and, for every $j \in N_{\text{out}}^*(i)$, we have that $d_{j,\text{out}}(T^*) < d_{j,\text{out}}(T)$ and $d_{j,\text{in}}(T^*) = d_{j,\text{in}}(T)$.

The opposite of an influencer is a *listener* – an agent who (1) does not cut any links with other agents, but (2) others cut links with her, and (3) continues to be listened to only by other agents who satisfy (1) and (2) after confirmation bias has changed the network.

Definition 5. An agent i is a *listener* if $d_{i,\text{out}}(T^*) = d_{i,\text{out}}(T)$ and $d_{i,\text{in}}(T^*) < d_{i,\text{in}}(T)$; and, for every $j \in N_{\text{in}}^*(i)$, we have that $d_{j,\text{in}}(T^*) < d_{j,\text{in}}(T)$ and $d_{j,\text{out}}(T^*) = d_{j,\text{out}}(T)$.

A final restriction needed to understand shifts of influence at the individual level is to focus on a society that is “wise”, as defined by Golub and Jackson [2010]. Informally, a network is wise when each agent has a negligible influence on the outcome the society converges to. The following definition states this formally.

Definition 6. A large network T is *wise* if and only if $s_i \sim \mathcal{O}(\frac{1}{n})$ for all i , and therefore $s_i \approx 0$ as n gets large.

Armed with these definitions, we can characterize how confirmation bias changes the influence of influencers and listeners in a wise society.

Proposition 1. *Suppose T and T^* are large, wise networks, the influence of influencers rises due to confirmation bias and the influence of listeners declines due to confirmation bias.*

This result shows that confirmation bias increases the influence of a particular subgroup of agents – the influencers – who are embedded in a neighbourhood of other agents like them. Confirmation bias has the opposite effect on listeners. Given the restrictions imposed by definitions 4 and 5, a large network would typically have few (if any) influencers/listeners. An implication is that we are able to characterize changes in influence at the individual level only for a small subset of agents. Technically, the wisdom assumption allows us to ignore the effect on an agent’s influence of changes in the neighbours’ neighbours influence, which helps with tractability.

2.3 Polarization

A commonly held view is that our society is becoming increasingly polarized. Intuitively, confirmation bias may be a force that pushes society toward greater polarization by preventing communication between individuals with different views, and therefore delaying the process of finding common ground. Section 2.1 confirms this intuition in the long-run – a society without confirmation bias converges earlier than one with confirmation bias, and therefore is, by definition, less polarized in the gap between the two convergence times. This section shows that, subject to a mean-field assumption, confirmation bias causes society to be more polarized at each point in time.

There are numerous metrics in economics and other social sciences that capture the notion of polarization (e.g. Esteban and Ray [1994]). We adopt what is perhaps the most basic metric in a set-up with a continuum of beliefs.

Definition 7. The *polarization* of a society at time t is equal to $var(x_t) = \frac{1}{N} \sum_i (x_{i,t} - \mu)^2$.

While the asymptotic behavior of Markov systems is well-studied, it is notoriously challenging to make statements about their intermediate state. For tractability reasons, we assume that the distribution of initial signals is uniform, and also make the following mean-field assumption to reduce the level of noise in the system.

Assumption 1 (Mean-field). Each agent i 's neighborhood is representative of the society as a whole, so $\sum_{j \in N(i) \setminus \{i\}} x_{j0} \approx \mu$ for all $i \in N$ and $x_j \perp\!\!\!\perp T_{ij}$ for all i .

There are two justifications for using a mean-field approach. Theoretically speaking, it allows us to isolate the randomness introduced by confirmation bias from the intrinsic randomness of the system. The short and medium term evolution of beliefs will depend on the random initial allocation of signals. For instance, a handful of agents may stick to an extreme position for a while because they happen to form a close-knit community with similar beliefs to begin with. This particular instance of the evolution of the system will, therefore, have a high initial level of polarization that is unaffected by confirmation bias, but may make it more difficult to identify the effect of the bias on polarization. By assuming that each agent's neighborhood is representative of society as a whole, we block this channel – allowing us to explore whether confirmation bias on its own increases polarization.

In practice, the mean-field assumption is a reasonable approximation in a large society if we ignore the presence of homophily – the tendency to associate with like-minded people. In a large society, individuals have several friends and therefore the size of one's neighborhood ensures it is approximately a representation of the wider society. The simulations in section 5 show that the following result on polarization holds even if we drop the mean-field assumption.

Proposition 2. *Assume that $x_0 \sim U[0, 1]$ and the mean-field assumption holds. Then at each point in time polarization is (weakly) monotonically increasing in the strength of confirmation bias q .*

There are two steps to the proof. First, we show that, under the mean-field assumption, we can characterize the belief for each agent at time t as a weighted average between the initial belief and the average belief μ in society. Second, we prove that the variance of these beliefs is increasing in the strength of the self-loops, and, therefore, in the amount of confirmation bias.

2.4 Optimal networks

A benevolent social planner would want to maximize the chance that society converges to the truth. Confirmation bias works against the objective of the social planner in two ways. We examined the first one in section 2.2 – it redistributes the influence across agents so that some initial signals are weighted more than others. The second one is that confirmation bias may break the network into different components, which leads to information loss as the content of some initial signals is not aggregated. Throughout the paper we have ignored information loss by assuming that T^* is strongly connected, but in this section we relax this assumption to investigate the social planner's decision.

A benchmark case is an omniscient social planner who can observe the initial allocation of signals and the level of confirmation bias, and can then engineer the network. Appendix A.3 shows that in this case, if convergence to a consensus is possible, the planner can always guarantee society

converges to the truth by constructing an “octopus” network – an agent at the center who only listens to herself, and everyone else listening to the center directly or indirectly, depending on how far their signal is from that of the center.¹⁰ Convergence to the truth requires an octopus, rather than a simple star, network to prevent confirmation bias from breaking the network into separate components.

In a more realistic and interesting set-up, the social planner does not know the distribution of initial signals or the level of confirmation bias. We still assume, however, that she can specify the network structure T subject to a budget of $B \equiv nd$ links, where $d > 0$. The social planner can, therefore, specify the optimal network, and the following definition formalizes what optimality means.

Definition 8. A network T' is optimal given confirmation bias, q , and the budget of links, B , if:

$$T' = \operatorname{argmax}_{T \in \mathcal{T}} \{Pr(T^{*\infty} \cdot x_{i0} = \bar{x}_{i0}) \mid \#[T_{ij} > 0] \leq B\}$$

An optimal network maximizes the probability that society converges to the truth. The truth is the average of the initial signals, \bar{x}_{i0} , and is exactly the value a single Bayesian agent would reach after aggregating all of the initial signals. To reach the truth, society needs to incorporate *all* signals, and weight them all equally. Maximizing the probability of reaching the truth therefore requires minimizing the chance that one or more agents’ initial information is lost because the network breaks into multiple components. Notice that this condition does not depend on the size of the breakaway component(s) – the objective is to minimize *any* breakaway and not its size. Weighting all of the signals equally can then be achieved by network symmetry.

Before stating the main result of this section, we need one assumption to help with tractability.

Assumption 2 (Link independence). $Pr(T_{ij}^* \neq T_{ij} \mid T_{ik}^* \neq T_{ik}) = Pr(T_{ij}^* \neq T_{ij})$ for all k .

This assumption means we can ignore correlations among links that are removed due to confirmation bias. In particular, it assumes that the probability a link from i to j is removed does not depend on whether/how many other agents k listening to j have cut links. Clearly this is an approximation because it is more likely that links to an agent with an extreme belief will be removed, but this type of correlations are challenging to handle in a network context and putting them aside allows us to prove the following statement.

Proposition 3. *Assume link independence and a budget of $B \equiv nd$ links. Then T is optimal if it is symmetric, has no self-links, and its unweighted equivalent is vertex-transitive with degree d .*

As Remark 1 showed, symmetric networks ensure that every initial signal receives equal weight in the learning process. The absence of self-links ensures we allocate all links in the budget to

¹⁰In this octopus network, convergence will happen in at most $\frac{1}{q}(\max_i \{x_{i0}\} - \min_i \{x_{i0}\})$ periods.

increase the robustness of the network to breakaway components. Vertex transitivity means that the network is completely homogeneous so every agent is identical in the structure of their interactions. Therefore, by exhausting the available budget of links, this minimizes the probability of a single and/or set of agents breaking into a separate component. Notice that the statement does not constrain the distribution of link weights so the set of networks that are optimal is quite large. Once we ignore link weights, however, all these networks are a member of the small class of vertex transitive networks, which is a subset of regular networks.

Definition 8 focuses exclusively on the final outcome of getting to the truth, but one may argue that a social planner would also care about achieving this quickly. This would entail characterizing the distribution of link weights that maximize convergence speed in the class of optimal networks. To the best of our knowledge, this is an unsolved problem in the graph theory literature.¹¹ There are, however, algorithms to obtain an approximate characterization. Appendix A.3 discusses two well-known ones – the Maximum Degree Heuristic and the Metropolis-Hastings algorithm. Both suggest that, given the constraints imposed by Proposition 3, unweighted networks are likely to converge quickly. In other words, a social planner that also cares about the speed of convergence would engineer a network that is symmetric, unweighted and vertex transitive.

3 Shock elections

The previous section has shown that confirmation bias affects the process of learning from others by making it slower, more polarized and by redistributing individuals’ influence. In the past decade, the advent of social media has arguably increased the visibility and weight that information we learn from others has on our views, and this has become increasingly relevant in the context of elections (see, e.g., Kohut et al. [2008], Braha and De Aguiar [2017] and Weeks et al. [2017]). In order to examine how the impact of confirmation bias on learning affects electoral outcomes, we embed the learning framework in section 1 into a basic voting model. The objective is to understand whether “shock” elections are more likely in a world where learning is affected by confirmation bias.

We assume there are two candidates $Y = \{0, 1\}$ whose belief, or *ideology*, is fixed at $x_{Y=0} = 0$ and $x_{Y=1} = 1$. There are n voters and each voter has an initial belief $x_{i0} \in \{x_{EL}, x_{CL}, x_S, x_{CR}, x_{ER}\}$ with $x_{EL} < x_{CL} < x_S = \frac{1}{2} < x_{CR} < x_{ER}$. Denote by f_i the fraction of voters assigned initial belief x_i , with $0 < f_i < 1$ and $\sum_{i=EL}^{ER} f_i = 1$. For expository purposes, we can think of 0 as the “Left” candidate, and 1 as the “Right” candidate. Initially, the spectrum of voters’ preferences spans “Extreme Left” (EL), “Center Left” (CL), “Swing voters” (S), “Center Right” (CR), and “Extreme Right” (ER).

Voters communicate through a network T according to the model in section 1. Each voter i ’s utility function $U_{it} = u(|x_{it} - x_y|)$ is strictly decreasing in the distance between their own belief

¹¹Boyd et al. [2004] show that a convex optimization problem can find a solution numerically, but do not provide any common characteristics of the solution.

(at the time voting takes place) and the belief of the winning candidate. The winner of an election at time t is determined by simple majority – the candidate with the most votes wins and a tie is resolved by a coin toss. This set-up implies that sincere voting is a weakly dominant strategy by application of the standard Median Voter Theorem result.¹² Thus, a voter i facing an election at time t casts a vote $v_{i,t}$ according to the following strategy:

$$v_{i,t}(x_{i,t}) = \begin{cases} 0 & \text{if } x_{i,t} < 0.5 \\ 1 & \text{if } x_{i,t} > 0.5 \\ \xi & \text{if } x_{i,t} = 0.5 \text{ with } P(\xi = 0) = P(\xi = 1) = 0.5 \end{cases}$$

Without loss of generality, throughout this section we assume that $f_{EL} + f_{CL} > f_{CR} + f_{ER}$ and $\sum_{i=EL}^{ER} x_{i0} f_i < 0.5$. The first assumption guarantees that if an election were to happen at time $t = 0$ before any learning takes place then the Left would win it. The second assumption states that a society which correctly aggregates all initial information would in the end vote for the Left as well. In order to focus our attention on the interesting case in which the learning process converges to the truth, we further assume that the society is “wise” as defined in Definition 6.

These assumptions restrict our attention to a society that votes for the same candidate – the Left one without loss of generality – before learning takes place and after learning has occurred. The outcome we are interested in is whether the presence of confirmation bias makes the shock electoral results of the Right candidate winning more likely.

Definition 9. In a society that exhibits wisdom, a *shock election* occurs if there exists a time $0 < t < \infty$ such that the Right wins the election that occurs at t .

The following proposition shows that confirmation bias makes a shock electoral outcome possible even in a society that exhibits wisdom.

Proposition 4. *In a large society where the mean-field and wisdom assumptions hold, a shock election can occur with confirmation bias, but it never occurs without confirmation bias.*

As discussed in section 2.3, the purpose of the mean-field assumption is to isolate the randomness introduced by confirmation bias from the intrinsic stochasticity of the system. Without the mean-field assumption, a shock election may occur even without confirmation bias simply because beliefs have a random fluctuation toward the Right at some point in the learning process before converging to vote for the Left. The proof in Appendix B.5 shows that this does not happen with the mean-field assumption, and this provides a clear benchmark to investigate whether confirmation bias adds additional noise that may cause a shock election. Furthermore, the mean-field assumption is

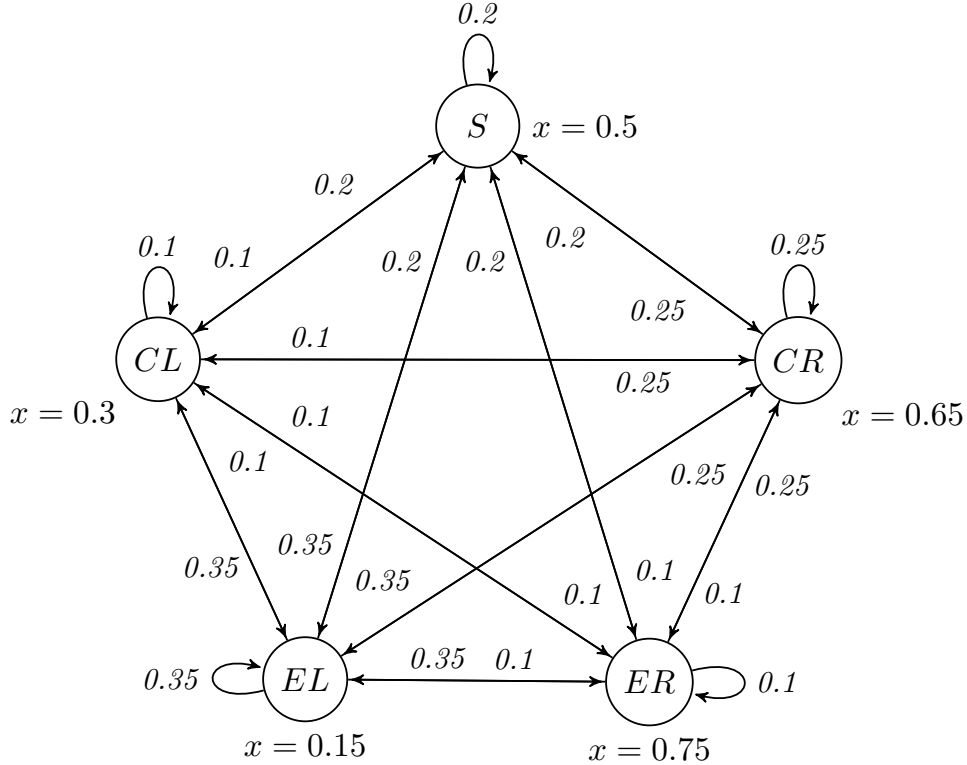
¹²In particular, our set-up satisfies the following conditions: (i) there are only two candidates; (ii) candidates are chosen by a majority vote; (iii) candidates have single peaked preferences; and (iv) voting takes place along a single dimension. See Downs [1957], and Mas-Colell et al. [1995, Chapter 21.D] for a review of the Median Voter Theorem.

a good approximation in large networks and the simulations in section 5 show that the result in Proposition 4 is robust to relaxing this assumption.

We prove that a shock election can occur with confirmation bias with the counterexample in Figures 3 and 4. Consider a society with 5 voters connected by network T represented in Figure 3. Suppose initial beliefs are captured by x_0 below, it is straightforward to compute the learning process.

$$T = \begin{pmatrix} 0.35 & 0.1 & 0.2 & 0.25 & 0.1 \\ 0.35 & 0.1 & 0.2 & 0.25 & 0.1 \\ 0.35 & 0.1 & 0.2 & 0.25 & 0.1 \\ 0.35 & 0.1 & 0.2 & 0.25 & 0.1 \\ 0.35 & 0.1 & 0.2 & 0.25 & 0.1 \end{pmatrix} \quad x_0 = \begin{pmatrix} 0.15 \\ 0.3 \\ 0.5 \\ 0.65 \\ 0.75 \end{pmatrix} \quad \Rightarrow \quad T \cdot x_0 = \begin{pmatrix} 0.42 \\ 0.42 \\ 0.42 \\ 0.42 \\ 0.42 \end{pmatrix} = T^\infty \cdot x_0$$

Figure 3: Network T , without confirmation bias. Notice that links are drawn with a double arrow to avoid cluttering, but the strength of the link differs depending on the direction and it is indicated in *italics* next to the arrow heads.

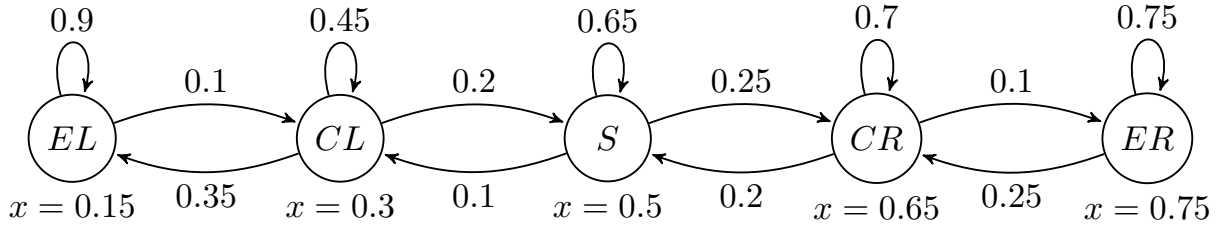


Clearly the society votes for the Left candidate at time 0, time 1, and every period thereafter. Now suppose that the level of confirmation bias is $q = 0.78$. Figure 4 represents the resulting network T^* after the removal of links. If we compute the learning process then, by the wisdom

assumption, it converges to the same outcome as without confirmation bias, but now a shock election occurs at time $t = 1$ when the Right would win.

$$T^* = \begin{pmatrix} 0.9 & 0.1 & 0 & 0 & 0 \\ 0.35 & 0.45 & 0.2 & 0 & 0 \\ 0 & 0.1 & 0.65 & 0.25 & 0 \\ 0 & 0 & 0.2 & 0.7 & 0.1 \\ 0 & 0 & 0 & 0.25 & 0.75 \end{pmatrix} \Rightarrow T^* \cdot x_0 = \begin{pmatrix} 0.165 \\ 0.2875 \\ 0.5175 \\ 0.63 \\ 0.725 \end{pmatrix} \Rightarrow \dots \Rightarrow T^{*\infty} \cdot x_0 = \begin{pmatrix} 0.42 \\ 0.42 \\ 0.42 \\ 0.42 \\ 0.42 \end{pmatrix}$$

Figure 4: Network T^* , with confirmation bias



What the counterexample shows is that a shock election can occur in a society that would otherwise make the correct decision if there was no learning and/or no confirmation bias. In particular, the effect of confirmation bias is that it can push a majority to support the Right candidate in the short-term even if in the long-term the society correctly aggregates the initial signals and votes for the Left candidate. In this counterexample, this happens because of two effects: (i) Swing voters stop listening to the Extreme Left/Right voters due to confirmation bias, and (ii) they put more weight on the views of Center Right compared to Center Left voters. This means that in the short-term the majority of Swing voters vote for the Right candidate who obtains a majority support. In the long-term, learning continues, leading to the correct aggregation of the initial information thanks to the wisdom assumption, but this means that there is a time window when a shock election can occur which would not happen in a world without confirmation bias.

4 Fringe media organizations

Changes in technology and the emergence of social media have made it easier to find news and information tailored exactly to your preferences [Sunstein, 2018, Chapter 3]. This has been accompanied by a polarization of the landscape, with outlets characterized by an extreme ideology achieving a large audience (Prior [2013] provides some support for this view). The purpose of this section is to examine the role of confirmation bias in the growth of media organizations with extreme ideologies. To do this, we extend the model in section 1 by introducing media organizations which choose their ideology and care about maximizing their audience, and by having agents form a link with a media organization before learning takes place.

Consider a media organization $m \in M \setminus \{1, \dots, M\}$ and assume that $M > 5$.¹³ Each media organization has utility $V_m(\mu_m) = \#\{i \in N : T_{im} > 0\}$ – it only cares about the number of agents who listen to it. At time $t = -1$ each m picks an ideology $\mu_m \in [0, 1]$ such that $\max_{\mu_m \in [0, 1]} V_m$ knowing that the initial belief of agent i is a random draw from $X \sim U[0, 1]$. Once chosen, the ideology of each media organization m is fixed, so we can represent each m as a node with a self-link of weight 1.¹⁴

As in the set-up in section 1, at time $t = 0$ each agent is endowed with a signal $\theta_i \sim U[0, 1]$ which is agent i 's initial belief x_i .¹⁵ The novel element here is that each agent also forms a link of weight ϵ with one and only one media organization at zero cost. Both the assumptions that the link weight is ϵ and it is costless are for simplicity given that the focus of this section is on the choice of ideology by the media organizations.¹⁶ We assume each agent i 's payoff is equal to $U_i = U_i(|x_{i0} - \mu_m|)$ with $\frac{\delta U_i}{\delta |x_{i0} - \mu_m|} < 0$, i.e. agents choose to listen to the media organization whose belief is closest to their own. If there is confirmation bias of strength q then an agent i forms a link with a media organization m only if $|x_{i0} - \mu_m| \leq 1 - q$.

Let Ψ_M^q be the set of equilibria with M firms and confirmation bias of strength q .¹⁷ An equilibrium $\psi = \{\mu_1, \dots, \mu_M\} \in \Psi_M^q$ is a set of ideologies for the media organizations. The focus of our analysis is the media organization with the most extreme ideology $\mu_{fr}(q, M) = \min_{\mu_m \in \Psi_M^q} \{\mu_m\}$ which we dub the “fringe” media organization. Notice that, by the symmetry of our set-up, it suffices to investigate the most extreme Left ideology because for any set of ideologies that constitutes

¹³The $M > 5$ assumption allows us to focus on the interesting and relevant case of a competitive media market. Appendix A.4 solves the model for the $M \leq 5$ case.

¹⁴This is equivalent to Hotelling [1929]'s model with fixed prices where the $[0, 1]$ line represents the unidimensional ideology space. Eaton and Lipsey [1975] presents results for this model in the absence of confirmation bias. Anand et al. [2007] examine media bias under the same assumption that media organizations care only about profits. In contrast to this model, they assume that agents care about some objective *truth*, as well as ideology.

¹⁵Notice that here we assume signals are drawn from the uniform distribution for analytical tractability.

¹⁶Picking a weight of ϵ allows us to sidestep the issue of how each agent redistributes the weights of their outgoing links to accommodate this new connection. The assumption that each agent forms one and only one link is for analytical tractability and can be justified behaviorally by limited attention (see, e.g., Gabaix [2014], and Masatlioglu et al. [2012] and references therein).

¹⁷An equilibrium always exists by standard existence results. For $M > 5$ there are infinitely many equilibria both with and without confirmation bias.

an equilibrium, its mirror image is also an equilibrium.

Our first result shows that the ideology of the fringe media organization is increasing in the competitiveness of the market.

Proposition 5. *The ideology of the fringe media organization becomes (weakly) more extreme as the number of media organizations M increases.*

The proof is an application of a result from Eaton and Lipsey [1975, p31], and is therefore omitted. The intuition is that media organizations try to spread out in ideology space to maximize the number of agents who listen to them, and therefore an increase in the number of outlets in the market pushes the most extreme organization closer to the boundary. This is in agreement with the initial observation that the proliferation of media outlets has led to a polarization of the media landscape. The following statement shows how the presence and strength of confirmation bias affects this polarization.

Proposition 6. *The ideology of the fringe media organization becomes (weakly) more extreme as the strength of confirmation bias q increases.*

The presence of confirmation bias reduces the incentive of a fringe media organization to moderate its ideology. Absent confirmation bias, the most extreme organization would want to moderate its ideology in order to attract more moderate listeners as this would not cost them any extreme listeners. What limits this moderation is the presence of other more moderate organizations that might want to ‘leapfrog’ the fringe media if it becomes too moderate. In the presence of confirmation bias, however, the fringe media organization does not want to moderate its ideology as much, for fear of losing some of its most extreme listeners.

5 Simulations

At different points in the paper, we rely on restricting the class of networks and/or a mean-field assumption to obtain analytically tractable results in this paper. For example, Theorem 1 only holds for symmetric networks where agents listen mostly to themselves. The proofs to Proposition 2 (polarization) and Proposition 4 (shock elections) require the mean-field assumption. In this section we run an extensive set of simulations to show that our results largely hold even after relaxing the mean-field assumption and apply to the general class of directed, weighted networks.

The first step of the simulations is to build a network with realistic structural features. We construct a modified version of the algorithm in Jackson and Rogers [2007], which creates networks that match the main structural characteristics of actual social networks. We start at step $k = 0$ from an initial cluster M of $m = 40$ nodes such that every node is connected to every other, with the direction of the link randomly determined with 50% probability; i.e. if $T_{ij} = 1$ then $T_{ji} = 0$ and if $T_{ij} = 0$ then $T_{ji} = 1$ for all $i, j \in M$. In step $k = 1$ a new node is introduced and randomly “meets”

$m_r = 20$ other nodes. In each random meeting there is a $p_r = 0.8$ probability of forming a link, with the direction of the link randomly determined with 50% probability. After these connections through random meetings have been formed, the new node meets $m_n = 20$ neighbors of her new connections. There is a $p_n = 0.8$ probability that she forms a link with each of these connections, and the direction of the link is randomly determined with 50% probability. In step $k = 2$ a new node is introduced and the process repeats itself. The network formation process stops at step $K = 960$ when we obtain a directed, weighted network formed by 1,000 agents.¹⁸ An agent listens to all of their neighbors equally, but the number of neighbors differs across agents. Therefore, link weights depend on the agent who is listening, and so the network is *weighted*.

In each set of simulations, we generate 1,000 different networks in this manner. For each of the networks, we execute the assignment of beliefs 100 times. This gives 100 instances to compare the outcomes of the learning process with and without confirmation bias for each of the 1,000 network structures.

In the first set of simulations, we test the generality and robustness of the results in section 2. After the network T is formed, each subject is assigned an initial belief randomly drawn from a uniform distribution $U[0, 1]$. Once initial beliefs are assigned, we randomly draw one value of confirmation bias q from the uniform distribution $U[0.05, 0.15]$. Starting from T , we remove any link T_{ij} where $|x_{i0} - x_{j0}| > 1 - q$ to form the network T^* . The simulation runs the learning process in our set-up on both T and T^* and we compare outcomes between the two networks.

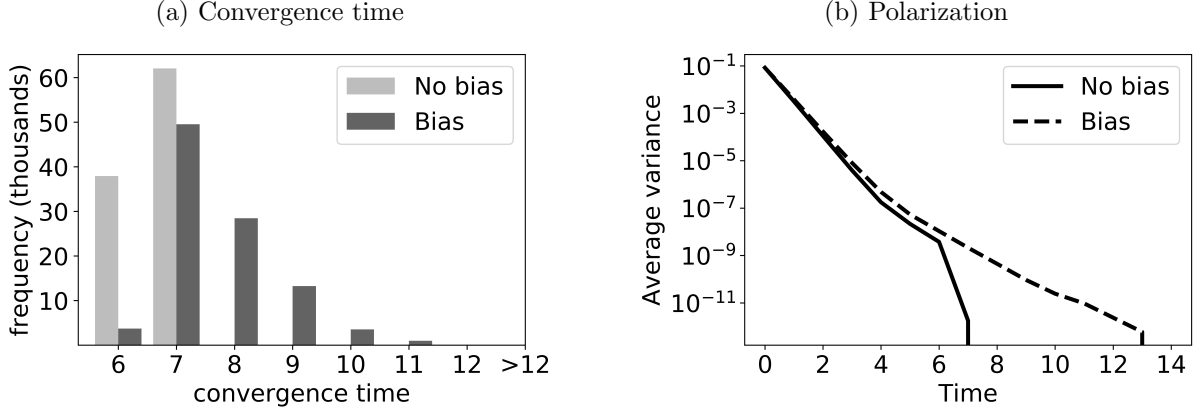
The histogram in the left panel of Figure 5 shows the frequency distribution of convergence time in T (light grey bars) and T^* (dark grey bars). From simple visual inspection, it is clear that the presence of confirmation bias shifts the distribution to the right, in line with Theorem 1. This is despite the fact that Theorem 1 assumes that *every* link is symmetric and that $T_{ii} \geq \frac{1}{2}$ for all i , while (by construction) *none* of the links in the simulated networks are symmetric and $T_{ii} = 0$ for all i – the result holds despite choosing the most challenging tests of the assumption made in the theory.¹⁹ The mean convergence time without confirmation bias is 6.6 periods (with $\{min, max\} = [6, 8]$), while it is 7.7 with confirmation bias ($\{min, max\} = [6, 26]$).

The right panel of Figure 5 shows the evolution over time of the levels of polarization averaged over the learning processes with (dashed line) and without confirmation bias (solid line). Before any learning takes place, the levels of polarization are identical, as the initial allocations of signals are the same. As soon as the learning process begins, however, the average level of polarization with confirmation bias is higher and it remains that way until everyone converges to the same belief. Given that the time to convergence is longer with confirmation bias, there is a period when the society has positive level of polarization with confirmation bias and a zero level in the absence of the bias. This shows the result in Proposition 2 is robust to relaxing the mean-field assumption.

¹⁸In case the resulting network is not strongly connected, we disregard it and restart the process to construct a new one.

¹⁹In 0.38% of cases, simulated convergence is one period faster with confirmation bias than without.

Figure 5: Convergence time with and without confirmation bias

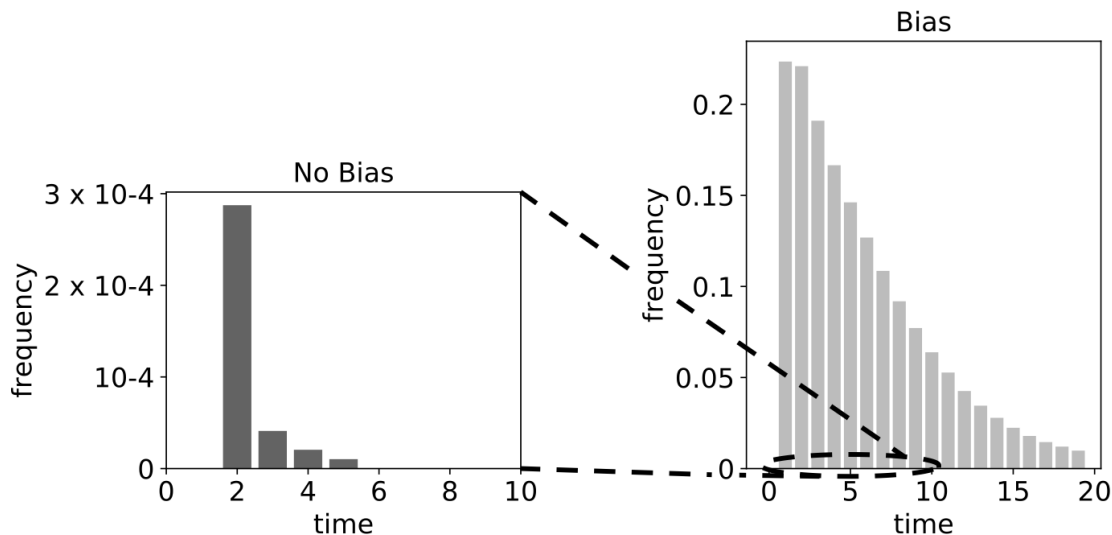


In the second set of simulations, we test the robustness of the results on voting in section 3 to relaxing the mean-field assumption. Following the set-up in section 3, we discretize the distribution of signals into $x_i \in \{x_{EL}, x_{CL}, x_S, x_{CR}, x_{ER}\} = \{0, 0.25, 0.5, 0.75, 1\}$.²⁰ Denote by f_i the fraction of agents assigned initial belief x_i . We fix $f_S = 0.2$ so that 20% of society is made up by swing voters. The distribution of initial beliefs of the rest of the society is randomly determined with $f_{EL} \sim U[0.1, 0.35]$, $f_{CL} = 0.45 - w_{EL}$, $f_{CR} \sim U[0.1, 0.25]$, $w_{ER} = 0.35 - f_{CR}$. Additionally, we keep taking random draws of f_{CR} until the weighted average of initial beliefs is less than 0.5. This set-up ensures we focus on the interesting case when society would vote for the Left before learning takes place and after learning has finished.

Assuming there is an election at each point in time, Figure 6 shows the fraction of shock elections with the Right winning both with (light grey bars) and without (dark grey bars) confirmation bias. The results clearly show that the statement of Proposition 4 holds even after relaxing the mean-field and wisdom assumptions. At its peak at time $t = 1$, 22.3% of elections with confirmation bias end up with the shock outcome of a win by the Right. In contrast, the maximum fraction of shock outcomes without confirmation bias is a paltry 0.03%, again at $t = 1$. Simulations show that the tiny fraction of shock elections without confirmation bias happen shortly after learning starts, while shock elections can occur in both the short and medium term with confirmation bias and well after the time of convergence without confirmation bias. This is closely related to the fact that convergence is relatively fast without confirmation bias, but can take a long time with it. In particular, the latest time of a shock election without confirmation bias in the 100,000 iterations was at $t = 5$, while, e.g., with confirmation bias at $t = 10$ we have that 6.4% of elections result in a shock outcome and there are still a few instances of shock outcomes when $t > 100$.

²⁰The beliefs do not need to be equally spaced on the line, but we choose this to aid exposition.

Figure 6: Wins by Right-wing candidate



6 Conclusion

The advent of online social networks has dramatically expanded the number of people who shape our views. Another facet of this change is that it has become very easy to stop listening to someone; it suffices to “unfollow” them. A large number of studies in psychology show that confirmation bias is a powerful filter in how we process information, and, inevitably, the relevance of this filter increases when it becomes so easy to ignore discordant voices. This paper has incorporated confirmation bias into a model of social learning on a network to investigate its impact on the learning process and the political arena.

Confirmation bias has an unambiguously negative impact. An increase in confirmation bias slows down learning and increases polarization in society. The type of network architectures that minimize these negative effects make the position of every individual interchangeable, and they are, unfortunately, the polar opposite of the networks we find in the real world. In the political context, the presence of confirmation bias makes it possible for the worse candidate (given the available information) to win in a shock election result. Moreover, confirmation bias makes the ideology of fringe media organizations more extreme.

This paper showcases the potential of combining insights from behavioral economics and networks. Aside from a few exceptions,²¹ there is a dearth of papers combining these two methodologies despite their common objective to incorporate realistic features of, respectively, the psychology of decision-making and social interactions into the standard economic framework. Our hope is that this contribution will encourage further studies at the intersection of these two literatures.

²¹See, e.g., Fryer and Jackson [2008] and Dessi et al. [2016].

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A Appendix: Extensions

This appendix contains generalizations and extensions to the main model.

A.1 Generalized model

The model in Section 1 assumes that the strength of confirmation bias is identical for all agents, and that confirmation bias causes agents to cut links *completely*. Additionally, it assumes that agents *only* cut links before the learning process begins. This section shows that the main result, Theorem 1 is robust to relaxing all three of these assumptions.

Relaxing these assumptions yields an updated definition of confirmation bias. First, the strength of confirmation bias, q , now depends on the agent, i . It allows arbitrary heterogeneity in the strength of confirmation bias. Second, confirmation bias no longer reduces link weights to zero, but rather reduces them by a link-specific factor $(1 - \alpha_{ij})$. There are no restrictions on the distribution of the α_{ij} 's (except of course that they are bounded between 0 and 1). Finally, the network can be changed every period, rather than only once – so there is now a sequence of networks T^{t*} .

The assumptions from Section 1 that agents never reinstate links, and that they redistribute the weight from cut/weakened links to themselves remain unchanged.

Definition A.1. A society on a network T in which agents have (agent-specific) confirmation bias $q_i \in (0, 1]$ communicates according to a network T^{t*} such that:

$$\begin{array}{lll} \text{if } |x_{it} - x_{jt}| > (1 - q_i) & \text{then} & T_{ij}^{t*} = (1 - \alpha_{ij})T_{ij}^{t-1*} & T_{ii}^{t*} = T_{ii}^{t-1*} + \alpha_{ij}T_{ij}^{t-1*} \\ & \text{otherwise} & T_{ij}^{t*} = T_{ij}^{t-1*} & T_{ii}^{t*} = T_{ii}^{t-1*} \end{array}$$

Where $q_i \in [0, 1]$, $\alpha_{ij} \in [0, 1]$, and $T_{ij}^{-1*} \equiv T_{ij}$ for all i and j .

Using this generalized definition, it is no longer immediately obvious what it means for confirmation bias to increase. We say that confirmation bias has increased if $q'_i \geq q_i$ and $\alpha'_{ij} \geq \alpha_{ij}$ for all i and j , with at least one strict inequality.

The proof to Theorem 1 shows that the *average convergence time*, τ , increases when link weight is moved from the between-agent links to agents' self-links (mathematically, this is moving weight from the off-diagonal elements to the diagonal elements of the matrix T) and there are *no other changes* to the network. It does not rely on any other structure to the changes to the network. Theorem 1 is therefore effectively proved as a special case, and the result holds under the generalization here without any change to the proof. This intuition is also unchanged – society converges to a consensus more slowly when agents are less willing to pay attention to those with different beliefs, and instead pay more attention to themselves.

In contrast, the influence, polarization and optimal networks results do not extend to this generalized setting.²² This is because they rely on the symmetric application of confirmation bias – if i cuts a link with j , then j must also cut a link with i (if a link is present in both directions). Both the heterogeneity in the strength of confirmation bias, q , and the heterogeneity in the down-weighting of links violate this requirement.

A.2 Speed of Learning

Theorem 1 uses a definition of convergence time, *average convergence time*, that does not explicitly involve the initial assignment of beliefs. However, it is robust to using an alternative measure; *consensus time* – which considers the time to converge to a consensus from the *worst-case* initial assignment of beliefs.

Definition A.2. [Golub and Jackson, 2010] The consensus time of the network T is:

$$CT(\epsilon) = \sup_{x_0 \in [0,1]^n} \min\{t : \|T^t x_0 - T^\infty x_0\|_{s(i)}^2 < \epsilon\}$$

where $\|\cdot\|_{s(i)}$ is a weighted version of the ℓ_2 norm.

The consensus time is the minimum time t at which all agents have beliefs within ϵ of each other in the *worst-case* distribution of initial signals. The difference is weighted by the influence, s_i , each agent has on the consensus. We can characterize confirmation bias' effect on consensus time for symmetric networks.

Theorem A.1. *If $T_{ii} \geq \frac{1}{2}$ for all i and T is symmetric, then the convergence time of T^* is weakly monotonically increasing in the amount of confirmation bias q .*

Before proving this result, it is helpful to state an existing result that links the consensus time to the second largest eigenvalue modulus (SLEM) of T . We use a result from Golub and Jackson [2008], although others exist.

Lemma A.1. [Golub and Jackson, 2008] *Assume T is connected, let λ_2 be the second largest eigenvalue of T and let s be the vector of influences with $\min_i s_i = \underline{s}$. If $\lambda_2 \neq 0$ then for any $0 < \epsilon \leq 1$:*

$$\left\lceil \frac{\log(1/4\epsilon) - \log(1/\underline{s})}{2\log(1/|\lambda_2|)} \right\rceil \leq CT(\epsilon) \leq \left\lceil \frac{\log(1/\epsilon)}{2\log(1/|\lambda_2|)} \right\rceil$$

Proof of Theorem A.1. The proof to Theorem 1 shows that all eigenvalues of the network are weakly positive and are weakly increasing in the strength of confirmation bias (in particular, see

²²The exception to this is Remark 2. This is because it considers a case where only one agent is affected by confirmation bias, so the heterogeneity in q and in α does not matter – as it has no effect.

the final paragraph of the proof). Therefore the second largest eigenvalue must also be the SLEM. Therefore, a weak increase in λ_2 implies a weak increase in the SLEM.

Lemma A.1 provides tight bounds for consensus time in terms $|\lambda_2|$ (the SLEM), and therefore shows that an increase in $|\lambda_2|$ weakly increases consensus time.

Taken together, stronger confirmation bias (weakly) increases $|\lambda_2|$, which in turn (weakly) increases the consensus time. \square

Notice that this uses the proof to Theorem 1 directly, and so relies on the same structure of changes to the network. Therefore, it also holds under the generalized model in Definition A.1.

Markov chain convergence is governed by the spectra of eigenvalues, and is often well approximated by the second largest eigenvalue modulus. In Theorem 1 we use the full spectra of eigenvalues, but use a metric that does not include the initial signals. This alternative result uses only the second eigenvalue as an approximation, but does include the initial signals. However, it only uses the worse-case set of signals. This has two drawbacks. First, we would like to know the convergence time for a given allocation of initial signals, x_0 , rather than only the worst-case allocation. Second, the allocation that constitutes the worst-case scenario for T may not also be the worst-case scenario for T^* . Therefore, this result does not guarantee that, for a given x_0 , consensus is reached more quickly under T than under T^* .

A.3 Optimal networks

This section first presents the benchmark case where the social planner is able to observe both the initial allocation of signals, x_0 , and the level of confirmation bias, q . It then introduces the Maximum-Degree Heuristic and the Metropolis-Hastings Heuristic – well-known heuristics for finding fast-converging Markov chains, and shows that they suggest unweighted networks as likely candidates for achieving fast convergence.

Optimal network with full information. We focus on the case where convergence to a consensus is possible. That is, where there exists $T \in \mathcal{T}$ such that T^* is strongly connected. This rules out the existence of a group $A \subset N$ where no agent $i \in X$ is willing to listen to any $j \in A^c$.

Since the social planner observes x_0 and q , she can then control the network T^* directly – as she knows exactly which links would, if created, be cut due to confirmation bias. Here, characterizing the optimal network is straightforward. It is a version of a ‘long-armed’ star network, which we call an *octopus* network. It guarantees convergence to the truth in at most $\frac{1}{q}(\max_i\{x_{i0}\} - \min_i\{x_{i0}\})$ periods.

To construct an octopus network, first identify an agent whose initial opinion is equal to the truth. Denote this agent a , and set $T_{aa} = 1$. If there does not exist such an agent, then identify a pair of agents, a and b where; (1) $|x_{a0} - x_{b0}| < 1 - q$ (they are willing to listen to one another), and (2) $\gamma x_{a0} + (1 - \gamma)x_{b0} = \bar{x}_{i0}$ for some $\gamma \in (0, 1)$ (some linear combination of their initial signals

is equal to the truth). Then choose $T_{aa}, T_{ab}, T_{ba}, T_{bb}$ such that x_{a1}, x_{b1} exactly equal the truth and $T_{aa} + T_{ab} = T_{ba} + T_{bb} = 1$.

Second, identify all agents i such that $0 \leq |x_{i0} - x_{a0}| < (1 - q)$, and set $T_{ia} = 1$. Denote this set of agents \mathcal{A}_1 . Next, identify all agents j such that $(1 - q) \leq |x_{j0} - x_{a0}| < 2(1 - q)$. Denote this set of agents \mathcal{A}_2 . Choose T_{ji} such that $\sum_{i \in \mathcal{A}_1} T_{ji} = 1$ and $\sum_{i \notin \mathcal{A}_1} T_{ji} = 0$. Repeat this step until $a \cup \bigcup_{k=1}^K \mathcal{A}_k = N$. Clearly $K \leq \frac{1}{q}(x_{max0} - x_{min0})$. So convergence to the truth is guaranteed within $\frac{1}{q}(x_{max0} - x_{min0})$ periods. Note that this process does not lead to a unique network – any network in the class of *octopus* networks constructed by this process is optimal.

Heuristics for link weights. Allocating link weights to minimize the time taken to reach a consensus is a well-studied problem in the mathematics literature.²³ Boyd et al. [2004] present a series of methods to solve this computationally for an individual network, but in the absence of general results we instead use heuristics to suggest link weights for optimal networks.

The Maximum Degree Heuristic assigns equal weight to *all* between-agent links in the network, chooses the maximum feasible link weight. All remaining weight is placed on the self-link. This means that the agent with the highest degree has no self-link.

Definition A.3 (Maximum Degree Heuristic). The maximum-degree transition probability matrix T^{md} is given by

$$T_{ij}^{md} = \begin{cases} 1/d_{max} & \text{if } (i, j) \in E \text{ and } i \neq j \\ 1 - d_i/d_{max} & \text{if } i = j \\ 0 & \text{if } (i, j) \notin E \end{cases}$$

Where E is the set of links, and $d_{max} = \max_i \{d_i\}$.

The Metropolis-Hastings Heuristic is somewhat more nuanced. For any pair of agents i and j , without loss of generality assume that i has a weakly higher degree. Then a link between i and j is assigned a weight equal to the *reciprocal of i 's degree*. This happens for all between-agent links. Self-links are then allocated to ensure that $\sum_j T_{ij} = 1$ for all i .

Definition A.4 (Metropolis-Hastings Heuristic). For a symmetric Markov chain, the Metropolis-Hastings transition probability matrix T^{mh} is given by

$$T_{ij}^{mh} = \begin{cases} \min\{\frac{1}{d_i}, \frac{1}{d_j}\} & \text{if } (i, j) \in E \text{ and } i \neq j \\ \sum_{(i,k) \in E} \max\{0, \frac{1}{d_i} - \frac{1}{d_k}\} & \text{if } i = j \\ 0 & \text{if } (i, j) \notin E \end{cases}$$

²³It is known as the Fastest Mixing Markov Chain problem, and the terminology used is rather different. Nevertheless, the core question is the same

Where E is the set of links.

Both of these definitions are taken from Boyd et al. [2004]. Due to the restrictions imposed by Proposition 3, we are only considering regular networks – implying that $d_i = d_j = d_{max}$ for all i . Therefore, under both the Maximum Degree and Metropolis-Hastings heuristics, all links in the network are assigned the same weight *and* there are no self-links.

A.4 Media

In this section we find the exact ideology of the fringe media organization for each value of $M \leq 5$, and for all values of $q \in [0, 1]$.

M=1: $\mu_{fr}(q, M = 1) \in [\min(q, 1 - q), \max(q, 1 - q)]$. Where there is a monopoly media organization, it chooses any ideology that will capture the largest possible audience (when $q \leq 0.5$ this encompasses all agents).

M=2: $\mu_{fr}(q \leq 0.5, M = 2) = 0.5$ and $\mu_{fr}(q > 0.5, M = 2) = 1 - q$. When $q \leq 0.5$, all agents are still willing to listen to an ideology of $\mu = 0.5$, and so the duopoly equilibrium is unaffected by confirmation bias in this range (see Eaton and Lipsey [1975] for a proof of the equilibrium). When $q > 0.5$ then the fringe media organization is willing to choose a sufficiently extreme ideology to attract the most extreme agent as a listener, *but no more*. Clearly a more extreme ideology would not be optimal, as the media organization could choose a more centrist ideology, and not lose any listeners on the fringe, but would gain some listeners to the center.

M=3: When $q \leq 0.75$ and there are three media organizations, an equilibrium does not exist. Therefore $\mu_{fr}(q \leq 0.75, M = 3)$ does not exist. $\mu_{fr}(q > 0.75, M = 3) = 1 - q$. With 3 media organizations and low confirmation bias, there is no equilibrium – peripheral organizations want to move inwards (towards the center), which increases their audience, but this causes the organization in the middle to want to change its ideology discontinuously to become a peripheral organization. This cycle never ends; preventing equilibrium existence. When $q > 0.75$ however, the peripheral organizations no longer want to move inwards past $\mu = 1 - q$ (or $\mu = q$), as they would lose very extreme agents from their audience by doing so. Given this, the organization in the middle no longer wants to switch its ideology to become a peripheral organization.

M=4 and M=5: $\mu_{fr}(q \leq 0.75, M = 4) = 0.25$ and $\mu_{fr}(q > 0.75, M = 5) = 1 - q$. $\mu_{fr}(q \leq \frac{5}{6}, M = 5) = \frac{1}{6}$ and $\mu_{fr}(q > \frac{5}{6}, M = 5) = 1 - q$. The logic here is identical to the two-organization case above.

We can see from this that Propositions 5 and 6 hold for two or more media organizations (aside from the existence issue when $M = 3$), but that neither hold for $M = 1$. At least some competition is required to sustain our results.

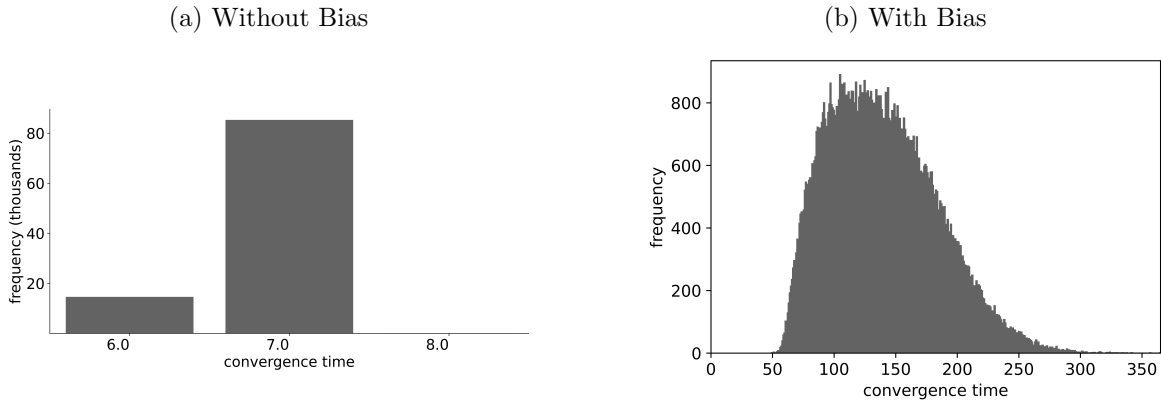
A.5 Simulations

In Section 5 we conducted two sets of simulations. In the first set, initial signals were drawn from a uniform distribution $U[0, 1]$, and confirmation bias was relatively low (between 0.05 and 0.15). This was used to test the robustness of Theorem 1 and Proposition 2. The second set of simulations, which used a discrete distribution of signals and a significantly higher value of confirmation bias, was used to test the robustness of Proposition 4.

This section uses the second set of simulations to further test the robustness of Theorem 1 and Proposition 2 – showing that they hold even more starkly under a higher level of confirmation bias. Note that it is not possible to use the first set of simulations to further test Proposition 4, as Proposition 4 requires restrictions on the distribution of initial signals not present in the first set of simulations.

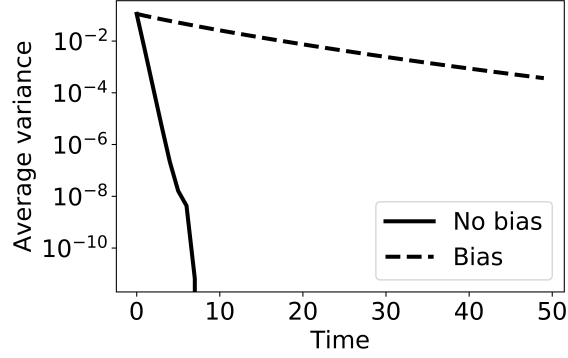
Figure 7 shows convergence times with and without confirmation bias. It is analogous to the left panel of Figure 5. Again, it is clear from visual inspection that confirmation bias shifts the distribution significantly to the right; in line with Theorem 1.²⁴ The mean convergence time without confirmation bias is 6.9 periods (with $\{min, max\} = [6, 8]$), while it is 139.3 with confirmation bias ($\{min, max\} = [49, 365]$). Figure 8 then shows polarization with and without confirmation bias. This is analogous to the right panel of Figure 5.

Figure 7: Convergence Speed



²⁴Recall that this is despite the fact that Theorem 1 assumes that every link is symmetric, while (by construction) none of the links in the simulated networks are symmetric – the result holds despite choosing the most challenging test of the assumption made in the theory.

Figure 8: Polarization

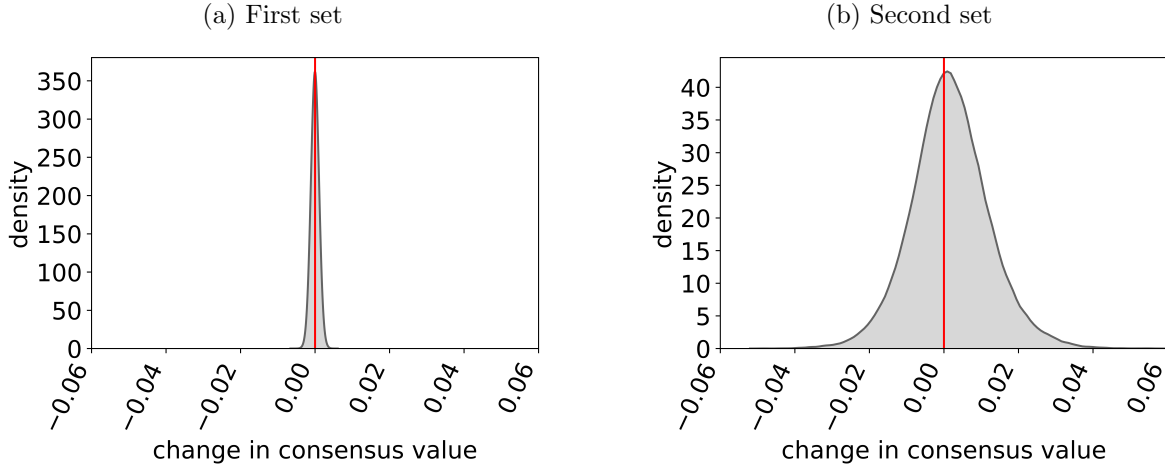


These confirm that Theorem 1 and Proposition 2 are robust to relaxing the network symmetry requirement and mean-field assumptions respectively. Additionally, they suggest that the results are in fact even more stark when confirmation bias is stronger.

Additionally, the simulations show an interesting feature of the changes in consensus values. Changes in consensus value (due to confirmation bias) appear to be symmetrically distributed about zero, and take on an approximate bell curve. Further, when confirmation bias is stronger, the changes are much more widely dispersed, but still centered around zero (see Fig 9).

Analytic results regarding the consensus value are intractable except in the special case of symmetric or wise networks, because it relies on the entirety of the influence vector and on the full distribution of initial signals. Relatively little is known about how elements of the influence vector change in response to changes in the network.

Figure 9: Consensus Change



B Proofs

We begin by setting out some notation. λ_k is the k th eigenvalue of the Markov chain T , where eigenvalues have been ordered such that $\lambda_1 > \lambda_2 > \dots > \lambda_n$. Similarly, λ_k^* is the k th eigenvalue of the Markov chain T^* . v_k is the right-hand eigenvector of the Markov chain T that corresponds to the eigenvalue λ_k . $\mathcal{E}(f, T)$ is the Dirichlet energy functional of T . s is the left-hand eigenvector that corresponds to the eigenvalue $\lambda_1 \equiv 1$, with i th element s_i . It is also called the *influence* vector, and s_i , the influence of agent i . f is an *arbitrary* vector of length n

B.1 Speed of Learning

Definition B.1. [Souzi, 2019, Definition 4.1] The Dirichlet energy function of a network T is

$$\mathcal{E}(f, T) = \frac{1}{2} \sum_{i,j} (f_i - f_j)^2 s_i T_{ij} \quad (\text{B.1})$$

Theorem B.1. [Souzi, 2019, Theorem 4.6]

$$1 - \lambda_j = \min_f \{ \mathcal{E}(f, T) : \|f\|_2 = 1, f \perp v_1, \dots, v_{j-1} \} \text{ for all } j \in \{1, \dots, n\} \quad (\text{B.2})$$

This states that 1 minus the j th largest eigenvalue equals the minimum of the Dirichlet energy (minimized w.r.t the argument f), subject to the constraint that the argument f is orthonormal to the $1^{st}, 2^{nd}, \dots, (j-1)^{th}$ eigenvectors of the matrix.

Lemma B.1. [Souzi, 2019, Lemma 3.5]

$$\|T^t(i, \cdot) - s\|_2^2 = \sum_{k=2}^n v_k(i)^2 \lambda_k^{2t} \quad (\text{B.3})$$

To avoid confusion, we use $v_k(i)$ to denote the i th element of the eigenvector v_k (which is itself the k th eigenvector of the matrix), and $T^t(i, \cdot)$ to denote the i th row of the matrix $(T)^t$.

Disambiguation: in this proof, we use subscripts to denote two different things. Subscripts on T , s , and f (i.e. T_{ij} , s_i , f_i) denote the i th (or ij th) element of the vector/matrix. Subscripts on v and λ (i.e. v_k , λ_k) denote the k th ordered eigenvector / eigenvalue. So v_k refers to a whole eigenvector (an $n \times 1$ vector), *not* a single element of it.

Proof of Theorem 1. Using Definition B.1 we have; $\mathcal{E}(f, T^*) = \frac{1}{2} \sum_{i,j} (f_i - f_j)^2 s_i^* T_{ij}^*$. Since T is symmetric, then $s_i^* = s_i$ for all i , so; $\mathcal{E}(f, T^*) = \frac{1}{2} \sum_{i,j} (f_i - f_j)^2 s_i T_{ij}^*$. By Definition 1; $T_{ij}^* \leq T_{ij}$ for all i, j with $i \neq j$. Terms where $i = j$ drop out of the summation, as $(f_i - f_j) = (f_i - f_i) = 0$. Therefore, $\mathcal{E}(f, T^*) \leq \mathcal{E}(f, T)$ for all f and any symmetric T and associated T^* .

Since $\mathcal{E}(f, T^*) \leq \mathcal{E}(f, T)$ for all f , then $\min_f \{\mathcal{E}(f, T^*)\} \leq \min_f \{\mathcal{E}(f, T)\}$ subject to any constraints on f . Therefore, it follows from Theorem B.1 that $1 - \lambda_j^* < 1 - \lambda_j$ for all j . This establishes that all eigenvalues are weakly monotonically increasing in q .

Simple manipulation of Lemma B.1 yields $\frac{1}{n} \sum_i \|T^t(i, \cdot) - s\|_2^2 = \frac{1}{n} \sum_{k=2}^n \sum_i v_k(i)^2 \lambda_k^{2t}$. By our normalizations, the squared elements of each eigenvector sum to 1; $(\sum_i v_k(i))^2 = 1$ for all k . Therefore $\frac{1}{n} \sum_i \|T^t(i, \cdot) - s\|_2^2 = \frac{1}{n} \sum_{k=2}^n \lambda_k^{2t}$. So we can write our convergence metric τ as;

$$\tau = \min \left\{ t > 0 : \frac{1}{n} \sum_{k=2}^n \lambda_k^{2t} < \epsilon \right\} \quad (\text{B.4})$$

It is clearly the case that $\frac{1}{n} \sum_{k=2}^n \lambda_k^{2t}$ is decreasing in t and increasing in λ_k for all $k \in \{2, \dots, n\}$. We know from earlier that $q' > q \implies \lambda'_k \geq \lambda_k$ for all $k \in \{2, \dots, n\}$. Therefore $q' > q \implies \frac{1}{n} \sum_{k=2}^n (\lambda'_k)^{2t} \geq \frac{1}{n} \sum_{k=2}^n \lambda_k^{2t}$. This follows from the fact that all eigenvalues are weakly positive. To see this, note that if $T_{ii} \geq \frac{1}{2}$ for all i , then there exists another Markov chain \tilde{T} such that $T = \frac{1}{2}(I + \tilde{T})$, where I is the identity matrix. All eigenvalues of I are equal to 1, and since \tilde{T} is a Markov chain, all of its eigenvalues are weakly greater than -1 .

So the minimum time taken to achieve $\frac{1}{n} \sum_{k=2}^n \lambda_k^{2t} < \epsilon$ is weakly monotonically increasing in q for any fixed $\epsilon > 0$. \square

Notice that this proof only relies on some off-diagonal elements of the matrix decreasing, and some corresponding diagonal elements increasing (and there being no other changes). Therefore, this proof clearly holds under the generalised model (set out in Definition A.1).

B.2 Influence

Proof of Remark 1. First, notice that if an $n \times n$ Markov chain, T , is symmetric, then it is doubly stochastic. This means that all rows and all columns sum to 1. Therefore, $\frac{1}{n} \cdot \mathbf{1} \cdot T = \frac{1}{n} \cdot \mathbf{1}$, where $\mathbf{1}$ is an $1 \times n$ vector of ones. Since $\lambda_1 \equiv 1$, then $\frac{1}{n} \cdot \mathbf{1}$ is the first left-hand eigenvector of T , which by definition is the influence vector.

Second, since $|x_{i0} - x_{j0}| \iff |x_{j0} - x_{i0}|$, in a symmetric network T , i cuts a link with j if and only if j cuts a link with i . This is due to homogeneity of q . Therefore, if T is symmetric, then T^* is also symmetric.

Taking these two observations together implies that $\frac{1}{n} \cdot \mathbf{1}$ is also the first left-hand eigenvector of T^* . Clearly, influences remain unchanged. \square

Proof of Remark 2. [Schweitzer, 1968, Section 3, equation 8] provides an equation for the influence of an agent changes following a perturbation in a single row: $s_i^* = s_i(1 + \frac{U_{ii}}{1 - U_{ii}})$, where $U = (T^* - T)Z$, and $Z = (I - T - \mathbf{1}s)^{-1}$ is the fundamental matrix of the Markov chain T .

When only agent i cuts links, then T^* differs from T only in the i th row, and so we can apply this formula. Further, when T^* differs from T only by entries in the i th row, then $U_{ii} =$

$(T_{ii}^* - T_{ii}) \sum_{j \neq i} (T_{ij}^* T_{ij}) Z_{ji}$. By construction; $\sum_j (T_{ij}^* - T_{ij}) = -(T_{ii}^* - T_{ii})$ and $(T_{ij}^* - T_{ij}) \leq 0$ for all $j \neq i$.

[Conlisk, 1985, Section 2, equation 8] shows that $Z_{ii} > Z_{ji}$ for all $j \neq i$. This implies that, $\sum_{j \neq i} (T_{ij}^* T_{ij}) Z_{ji} > 0$. Therefore $U_{ii} > 0$, and $s_i^* > s_i$.

Alternatively, we can see the same result by repeated application of the elementary perturbation result from [Conlisk, 1985, Section 4]. \square

Proof of Proposition 1. We prove the result for an *influencer*, i . Denote the set of influencers Θ , and the set of agents that influencers listen to $\hat{\Theta}$.

Taking the definition of influence for agent i ; $(1 - T_{ii})s_i = \sum_{j \in \mathbb{N}(i)} T_{ji}s_j$, and for agent j ; $(1 - T_{jj})s_j = \sum_{k \in \mathbb{N}(j) \setminus \{i\}} T_{kj}s_k + T_{ij}s_i$, and substituting the second into the first, yields;

$$(1 - T_{ii})s_i = \sum_{j \in \mathbb{N}(i)} T_{ji} \left(\left[\sum_{k \in \mathbb{N}(j) \setminus \{i\}} T_{kj}s_k T_{ij}s_i \right] (1 - T_{jj})^{-1} \right)$$

Rearranging yields;

$$(1 - T_{ii} - \left[\sum_{j \in \mathbb{N}(i)} T_{ji} T_{ij} (1 - T_{jj})^{-1} \right]) s_i = \sum_{j \in \mathbb{N}(i)} \sum_{k \in \mathbb{N}(j) \setminus \{i\}} T_{ji} T_{kj} s_k (1 - T_{jj})^{-1}$$

And by identical logic we can get an equivalent equation for s_i^* . By definition; (a) $i \in \Theta$, $T_{ji} = T_{ji}^*$ for all $j \neq i \in \mathbb{N}(i)$, and (b) $j \in \hat{\Theta}$, $T_{kj} = T_{kj}^*$ for all $k \neq j, i \in \mathbb{N}(j) \setminus i$. Now let (c) $T_{ii}^* = T_{ii} + \Delta_i$, $\Delta_i > 0$, and (d) $T_{jj}^* = T_{jj} + \Delta_j$, $\Delta_j > 0$. Substituting (a)-(d) into the equivalent equation for s_i^* , and then subtracting the equation for s_i yields;

$$\left(1 - T_{ii} - \Delta_i - \left[\sum_{j \in \mathbb{N}(i)} T_{ji} T_{ij}^* (1 - T_{jj} - \Delta_j)^{-1} \right] \right) s_i^* - \left(1 - T_{ii} - \left[\sum_{j \in \mathbb{N}(i)} T_{ji} T_{ij} (1 - T_{jj})^{-1} \right] \right) s_i = \left(\sum_{j \in \mathbb{N}(i)} \sum_{k \in \mathbb{N}(j) \setminus \{i\}} T_{ji} T_{kj} s_k^* (1 - T_{jj} - \Delta_j)^{-1} \right) - \left(\sum_{j \in \mathbb{N}(i)} \sum_{k \in \mathbb{N}(j) \setminus \{i\}} T_{ji} T_{kj} s_k (1 - T_{jj})^{-1} \right)$$

Rearranging the Right Hand Side (RHS) yields;

$$\text{RHS} = \sum_{j \in \mathbb{N}(i)} \sum_{k \in \mathbb{N}(j) \setminus \{i\}} \frac{T_{ji} T_{kj} (s_k^* - s_k)}{1 - T_{jj} - \Delta_j} + \sum_{j \in \mathbb{N}(i)} \sum_{k \in \mathbb{N}(j) \setminus \{i\}} \frac{\Delta_j T_{ji} T_{kj} s_k}{(1 - T_{jj})(1 - T_{jj} - \Delta_j)}$$

We assume that that third round effects are small; $T_{ji} T_{kj} (s_k^* - s_k) \approx 0$. This yields

$$\text{RHS} = \sum_{j \in \mathbb{N}(i)} \sum_{k \in \mathbb{N}(j) \setminus \{i\}} \frac{\Delta_j T_{ji} T_{kj} s_k}{(1 - T_{jj})(1 - T_{jj} - \Delta_j)}$$

By the definition of an *influencer*, if i cuts a link with j ($T_{ij}^* < T_{ij}$) then j cannot listen to i ($T_{ji} = 0$). Otherwise, j would cut her link with i – due to the homogeneity of confirmation bias – which is not permitted by the definition of an influencer. Therefore, $T_{ij}T_{ji} = T_{ij}^*T_{ji}$.

any agent who i cuts We know that in any instance where $T_{ij} \neq T_{ij}^*$, $T_{ji} = 0$, and so that component of the summation is zero. Therefore, it is appropriate to equate T_{ij}^* to T_{ij} in this summation, as any instances where they differed would be irrelevant (the summation component would be zero). Using this result, we can rearrange the Left Hand Side (LHS) to find;

$$\text{LHS} = \left(1 - T_{ii} - \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} \right) (s_i^* - s_i) - \Delta_i s_i^* - \sum_{j \in \mathbb{N}(i)} s_i \frac{(T_{ji}T_{ij})\Delta_j}{(1 - T_{jj})(1 - T_{jj} - \Delta_j)}$$

Aside: show that for $i \in \Theta$; $1 - T_{ii} - \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} > 0$. First, note that $T_{jj} + \Delta_j + \sum_i T_{jk}^* = 1$ by definition, and that $T_{ji}^* = T_{ji}$ for all $j \neq i \in \mathbb{N}(i)$. This implies that; $1 - T_{jj} - \Delta_j - \sum_{k \neq i \in \mathbb{N}(j)} T_{jk}^* = T_{ji}$. Since $\sum_{k \neq i \in \mathbb{N}(j)} T_{jk}^* \geq 0$, then; $1 - T_{jj} - \Delta_j \geq T_{ji}$. This rearranges to; $\frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} \leq T_{ij}$.

If $\frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} = T_{ij}$ for all j , then $1 - T_{ii} - \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} = 0$. However, there must exist at least one $j \in \mathbb{N}_i$ where $T_{ji} = 0$, as the definition of an influencer requires that i cuts links with some j , but no j cuts links with i . This in turn requires that there is some j who does not listen to i , but is listen to by i . Therefore;

$$1 - T_{ii} - \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} > 0$$

Now equate LHS and RHS (from above) and rearrange;

$$\left(1 - T_{ii} - \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} \right) (s_i^* - s_i) = \Delta_i s_i^* + \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}\Delta_j}{(1 - T_{jj})(1 - T_{jj} - \Delta_j)} \left(s_i T_{ij} + \sum_{k \in \mathbb{N}(j) \setminus \{i\}} T_{kj} s_k \right)$$

All Right Hand Side terms are weakly greater than zero, and we have established that $1 - T_{ii} - \sum_{j \in \mathbb{N}(i)} \frac{T_{ji}T_{ij}}{1 - T_{jj} - \Delta_j} > 0$. Therefore;

$$(s_i^* - s_i) \geq 0$$

The proof for a *listener* follows the same logic to that for an *influencer* and does not provide any additional insight. It is available from the authors upon request. \square

B.3 Polarization

Lemma B.2. *Under Assumption 1, the learning rule for an agent i can be expressed as $x_{it} = (1 - T_{ii}^t)\mu + T_{ii}^t x_{i0}$ where $\mu = \frac{1}{d_i} \sum_j x_{j0}$ for all j .*

Proof of Lemma B.2. Proof by induction. First, show for $t = 1$. By the definition of covariance; $cov(x_{j0}, T_{ij}) = \frac{1}{d_i} \sum_{j \in N(i) \setminus \{i\}} (x_{j0} - \bar{x}_{j0})(T_{ij} - \bar{T}_{ij})$. Which rearranges to;

$$d_i cov(x_{j0}, T_{ij}) = \left(\sum_{j \in N(i) \setminus \{i\}} x_{j0} T_{ij} \right) - \left(\bar{x}_{j0} \sum_{j \in N(i) \setminus \{i\}} T_{ij} \right) - \left(\bar{T}_{ij} \sum_{j \in N(i) \setminus \{i\}} x_{j0} \right) + d_i \bar{x}_{j0} \bar{T}_{ij}$$

Where $d_i = \#[T_{ij} > 0] = \#[N(i) \setminus \{i\}]$. By Assumption 1 we have $\bar{x}_{j0} \approx \mu$, and $cov(x_{j0}, T_{ij}) \approx 0$. By definition, $\bar{T}_{ij} = \frac{1}{d_i}(1 - T_{ii})$. Therefore, the equation above rearranges to; $\sum_{j \in N(i) \setminus \{i\}} x_{j0} T_{ij} - \mu(1 - T_{ii}) - \frac{1}{d_i}(1 - T_{ii})d_i\mu + d_i\mu\frac{1}{d_i}(1 - T_{ii}) \approx 0$, which in turn yields, $\sum_{j \in N(i) \setminus \{i\}} x_{j0} T_{ij} \approx \mu(1 - T_{ii})$. The learning rule in Section 1, can be decomposed to $x_{i1} = \sum_{j \in N(i) \setminus \{i\}} T_{ij}x_{j0} + T_{ii}x_{i0}$. Substituting our result from immediately above into the learning rule yields; $x_{i1} \approx \mu(1 - T_{ii}) + T_{ii}x_{i0}$, as required.

Now we assume that for $t = k$; $x_{ik} = (1 - T_{ii}^k)\mu + T_{ii}^k x_{i0}$, and prove for $t = k + 1$. The learning rule for $t = k + 1$ is; $x_{i,k+1} = \sum_{j \in N(i) \setminus \{i\}} T_{ij}x_{jk} + T_{ii}x_{ik}$. Substituting in our assumption for $t = k$, and multiplying out and rearranging terms;

$$x_{i,k+1} = \mu \left(\sum_{j \in N(i) \setminus \{i\}} T_{ij} \right) + \left(\sum_{j \in N(i) \setminus \{i\}} (x_{j0} - \mu) \cdot T_{ij} T_{jj}^k \right) + (T_{ii}\mu - T_{ii}^{k+1}\mu + T_{ii}^{k+1}x_{i0})$$

Recall that the covariance equation rearranges to $Ncov(a, b) = \sum^N ab - N\bar{a}\bar{b}$ for general a, b . Let $a = (x_{j0} - \mu)$, $b = T_{ij}T_{jj}^k$, $N = d_i$. By Assumption 1 $\bar{a} = 0$, so $d_i \cdot cov((x_{j0} - \mu), T_{ij}T_{jj}^k) = \sum (x_{j0} - \mu)T_{ij}T_{jj}^k$. Adding/subtracting a constant does not alter covariance, and a multiplicative constant acts linearly, so; $d_i T_{jj}^k cov(x_{j0}, T_{ij}) = \sum (x_{j0} - \mu)T_{ij}T_{jj}^k$. By Assumption 1, the covariance term approximately equals zero, so $\sum (x_{j0} - \mu)T_{ij}T_{jj}^k \approx 0$. Using this observation, the equation above simplifies to

$$x_{i,k+1} = \mu \sum_{j \in N(i) \setminus \{i\}} T_{ij} + (T_{ii}\mu - T_{ii}^{k+1}\mu + T_{ii}^{k+1}x_{i0})$$

And simple rearranging yields $x_{i,k+1} = \mu(1 - T_{ii}^{k+1}) + T_{ii}^{k+1}x_{i0}$. This completes the proof. \square

Proof of Proposition 2. Recall from Definition 7 that $var(x_t) = \frac{1}{N} \sum_i (x_{it} - \mu)^2$. Substituting in the simplified learning rule from Lemma B.2 and rearranging terms yields; $var(x_t) = \frac{1}{N} \sum_i T_{ii}^{2t} \cdot (x_{i0} - \mu)^2$. We now show that the belief at time t is always further from the mean in the case with confirmation bias.

Proof by induction. For $t = 1$; $x_{i1}^* = \sum_j x_{j0}T_{ij}^* + T_{ii}^*x_{i0}$, which rearranges to; $x_{i1}^* = (1 - T_{ii}^*)\mu + T_{ii}^*x_{i0} + Cov(x_{j0}, T_{ij}^*)$. Assumption 1; $Cov(x_{j0}, T_{ij}) = 0$, coupled with $x_i \sim U[0, 1]$ implied that $Cov(x_{j0}, T_{ij}^*)$ has the same sign as $x_{i0} - \mu$. If $x_{i0} > 0.5$, then after cutting links due to confirmation bias, i listens *disproportionately* to other agents with $x_{j0} > 0.5$. Conversely for $x_{i0} < 0.5$. Therefore, $|x_{i1}^* - \mu| - |x_{i1} - \mu| = |Cov(x_{j0}, T_{ij}^*)| > 0$

For $t = k$; assume that $x_{ik}^* = (1 - T_{ii}^{*k})\mu + T_{ii}^{*k}x_{i0} + \epsilon_{ik}$, where $\epsilon_{ik} = T_{ii}^{*k}\epsilon_{ik-1} + \text{Cov}(x_{jk-1}, T_{ij}^*)$ takes the same sign as $(x_{ik}^* - \mu)$.

For $t = k+1$; $x_{ik+1}^* = \sum T_{ij}^* x_{jk}^* + T_{ii}^{*k+1}x_{i0}$. First, we use a covariance expansion, then substitute in our assumption for x_{ik}^* . Next, we multiply out, using the covariance expansion and the assumption that $\sum x_{j0} = \mu$. Finally, we note that; (1) $\text{Cov}(T_{jj}^*, x_{j0}) = 0$ due to the mean field assumption and the uniform distribution of initial signals, and (2) $\sum \epsilon_{jk} = 0$ due to uniform distribution of initial signals, and therefore the symmetry of the problem. Notice that for a given agent i , $x_{it}^* - \mu$ has the same sign for all t . Therefore, $\text{Cov}(x_{jt}, T_{ij}^*)$ has the same sign as $x_{it}^* - \mu$, by identical logic to the $t = 0$ case. By defining $\epsilon_{jk+1} = T_{ii}^{*k}\epsilon_{jk} + \text{Cov}(x_{jk}, T_{ij}^*)$, we get the final equation, completing the induction.

$$\begin{aligned}
x_{ik+1}^* &= \sum T_{ij}^* \sum x_{jk}^* + T_{ii}^{*k+1}x_{i0} + \text{Cov}(x_{jk}, T_{ij}^*) \\
x_{ik+1}^* &= \sum T_{ij}^* \sum [(1 - T_{jj}^{*k})\mu + T_{jj}^{*k}x_{j0} + \epsilon_{jk}] + T_{ii}^{*k+1}[(1 - T_{ii}^{*k})\mu + T_{ii}^{*k}x_{i0} + \epsilon_{ik}] + \text{Cov}(x_{jk}, T_{ij}^*) \\
x_{ik+1}^* &= (1 - T_{ii}^{*k+1})\mu \sum (1 - T_{jj}^{*k}) + \mu \sum T_{jj}^{*k} + \text{Cov}(T_{jj}^*, x_{j0}) + \sum \epsilon_{jk} + T_{ii}^{*k+1}\mu - T_{ii}^{*k+1}\mu \\
&\quad + T_{ii}^{*k+1}x_{i0} + T_{ii}^{*k}\epsilon_{jk} + \text{Cov}(x_{jk}, T_{ij}^*) \\
x_{ik+1}^* &= (1 - T_{ii}^{*k+1})\mu + T_{ii}^{*k+1}x_{i0} + T_{ii}^{*k}\epsilon_{jk} + \text{Cov}(x_{jk}, T_{ij}^*) + (1 - T_{ii}^{*k})\text{Cov}(T_{jj}^*, x_{j0}) + \sum \epsilon_{jk} \\
x_{ik+1}^* &= (1 - T_{ii}^{*k+1})\mu + T_{ii}^{*k+1}x_{i0} + \epsilon_{jk+1}
\end{aligned}$$

Therefore, $|x_{ik+1}^* - \mu| - |x_{ik}^* - \mu| = |\epsilon_{jk+1}| > 0$.

Higher q increases the magnitude of $\text{Cov}(x_{j0}, T_{ij}^*)$, it weakly increases the average opinion that i continues to listen to if $x_{i0} > 0.5$ (and decreases that average opinion if $x_{i0} < 0.5$). In the special case where $x_{i0} = 0.5$, then the average opinion that i continues to listen to is unaffected by q . The symmetry of the problem in this case means that $\text{Cov}(x_{j0}, T_{ij}^*) = 0$ if $x_{i0} = 0.5$. By similar logic, it also increases the magnitude of $\text{Cov}(x_{jt}, T_{ij}^*)$ for all t . That variance is increasing in q follows from this observation. \square

B.4 Optimal networks

Lemma B.3. *Given a network, T , and vector of unallocated beliefs, x . Taking expectations over the realizations of (allocated) initial beliefs, the probability of cutting a randomly selected in T link is; (1) the same for all links, and (2) weakly monotonically increasing in q .*

Proof of Lemma B.3. Consider an arbitrary link, T_{ij} . The probability of cutting this link is the number of pairs of beliefs further apart than $(1 - q)$, as a fraction of the total number of pairs of

beliefs.

$$f_i(q) = Pr(\text{reroute}|q) = \frac{\#\{x_k, x_l : |x_k - x_l| > (1 - q), k \neq l\}}{\#\{x_k, x_l : k \neq l\}}$$

This did not depend on our choice of link T_{ij} , and so is the same for all links. Hence $f_i(q) = f(q)$ for all i . $\#\{x_k, x_l : |x_k - x_l| > (1 - q), k \neq l\}$ is weakly monotonically increasing in q , and $\#\{x_k, x_l : k \neq l\}$ is unaffected by q , so $f(q)$ is weakly monotonically increasing in q . \square

Definition B.2. Information loss, L , is the number of agents who cease to have influence in the giant component due to confirmation bias. $L = \#\{i : s_{\text{giant},i}^* = 0, s_{\text{giant},i} > 0\}$ Where $s_{\text{giant},i}$ is the influence of agent i in the giant component.

Definition B.3 (Vertex-transitivity). A network T is vertex-transitive if for any pair of agents i and j there exists a Graph Automorphism $\pi : N \rightarrow N$ such that $\pi(i) = j$

Proof of Proposition 3. First, some machinery. **Sets.** Let $g_{k,z}$ denote the z^{th} group of $k \in \{1, 2, \dots, n\}$ agents. Let $g_{k,z}^c$ be its complement. Note that $g_{k,z}^c \equiv g_{n-k,z}$. Let G_k denote the set of all groups of agents of size k . $|G_k| = Z_k$; G_k contains Z_k different groups. Let A_k denote the event that *any* one or more groups of size k become disconnected from the network due to confirmation bias. **Probabilities.** Let $Pr(A_k)$ is the probability that the event A_k occurs, and let $Pr(g_{k,z})$ be the probability that the z^{th} group of size k becomes disconnected. Let $Pr(L > 0)$ denote the probability that there is some information loss in the network. **Degrees.** Let $d_{g_{k,z}}$ denote the number of links between the agents in the group $g_{k,z}$ and those in $g_{k,z}^c$.

If T is symmetric and there is no information loss, then $T^* \cdot x_{i0} = \bar{x}_{i0}$ (by Remark 1). The optimal network is therefore a symmetric network that minimizes $Pr(L > 0)$. By Lemma B.3, the probability of cutting a randomly chosen link is equal to $f(q)$. Under Assumption 2 we apply this to all links, and ignore correlations between cutting probabilities. The probability a group $g_{k,z}$ becomes disconnected is the probability that all of their links to the rest of the network are cut. That is; $Pr(g_{k,z}) = f^{d_{g_{k,z}}}$. Then the probability that one or more group of size k becomes disconnected is one minus the probability that no groups of size k become disconnected. That is;

$$Pr(A_k) = 1 - \prod_{z=1}^{Z_k} [1 - Pr(g_k)] = 1 - \prod_{z=1}^{Z_k} [1 - f^{d_{g_k}}]$$

Due to the convexity of taking exponents, this term is minimized by setting $d_{g_{k,z}} = d_k$ for all $g_{k,z} \in G_k$, for any value of k . The overall probability of information loss, $Pr(L > 0)$, is increasing in $Pr(A_k)$ for all k . So a graph where all groups of agents of a given size have the same number of links minimizes the probability of information loss. This requires that all nodes are identical in the unweighted equivalent T^s . By definition, this is equivalent to the network T^s being vertex

transitive by definition. Regularity follows from vertex-transitivity, and the absence of self-links increases $Pr(g_k)$ for all g_k . \square

B.5 Shock elections

Proof of Proposition 4. First, recall that $\mu < 0.5$ and $\#[x_{i0} < 0.5] > \#[x_{i0} > 0.5]$ by assumption. The Left wins immediately and in the long run.

By Lemma B.2, when $q = 0$ we have the simplified learning rule; $x_{it} = (1 - T_{ii}^t)\mu + T_{ii}^t x_{i0}$ for all i . Let $\Delta x_{it} = x_{it} - x_{it-1}$ for $t \geq 1$. Substituting in the simplified learning rule and rearranging yields; $\Delta x_{it} = T_{ii}^{t-1}(1 - T_{ii})(\mu - x_{i0})$. t only enters as the exponent on a weakly positive number.²⁵ Therefore, the sign of $\Delta x_{i,t}$ is the same for all i, t .

Therefore, $(x_{it} - \mu)$ is weakly monotonically decreasing in t , for all i . This means that no agent who votes for the Left-wing candidate can ever change their vote (and all agents who vote for the right-wing candidate must switch their vote to the Left-wing candidate at some point). Therefore the Left-wing candidate would win for any t , and so there cannot be any volatility.

To prove that volatility is possible with $q > 0$ we show sufficient conditions for the Right-wing candidate to win at $t = 1$. Since the Left-win candidate wins at $t = 0$ and $t = \infty$ by assumption, this is sufficient to prove that volatility is possible.

Choose $q \in (1 - \max\{k_{m+1} - k_m\}, 1 - \min\{k_{m+2} - k_m\})$. Then we have;

$$\begin{aligned} x_{EL1} &\approx f_{CL}x_{CL0} + (1 - f_{CL})x_{EL0} \\ x_{CL1} &\approx f_{EL}x_{EL0} + f_Sx_{S0} + (1 - f_{EL} - f_S)x_{CL0} \\ x_{S1} &\approx f_{CL}x_{CL0} + f_{CR}x_{CR0} + (1 - f_{CL} - f_{CR})x_{S0} \\ x_{CR1} &\approx f_Sx_{S0} + f_{ER}x_{ER0} + (1 - f_S - f_{ER})x_{CR0} \\ x_{ER1} &\approx f_{CR}x_{CR0} + (1 - f_{CR})x_{ER0} \end{aligned}$$

Now suppose that $f_{EL} + f_{CL} < 0.5$ (the Left do not form a majority on their own) and that $f_{CR}x_{CR0} + f_{CL}x_{CL0} > 0.5$ (the center right can persuade the swing voters more than the center left). Then $x_{S1} > 0.5$, and the Right-wing candidate wins at $t = 1$. \square

B.6 Media

Proof of Remark 6. From Eaton and Lipsey [1975] we have; $\mu_{fr}(q = 0, M) = \frac{1}{2M-4}$. When $(1 - q) > \frac{1}{2M-4}$ then a fringe media organization does not lose any (potential) agents due to confirmation bias. For $q \in [0, \frac{2M-5}{2M-4}]$, the fringe media organization is unaffected by confirmation bias, and so its ideology is unaffected by confirmation bias in this range.

²⁵In the special case where $T_{ii} = 0$, agent i will arrive at the long run social consensus after only one period of learning. This is because, by assumption, their neighbors are representative of society as a whole.

Further, $\mu_{fr}(q, M) = (1 - q)$ when $q \in [\frac{2M-5}{2M-4}, 1]$. $\mu_{fr}(q, M) > (1 - q)$ is not possible. The fringe media organization can shift leftwards and pick up at least as many listeners to its left as it would lose to its right, meaning that it weakly prefers to shift to the left. Since this leftward shift does not reduce the audience of any other media organization, then it remains an equilibrium. This logic (implicitly) requires that the distribution of agents' initial opinions is uniform (or close to uniform).

$\mu_{fr}(q, M) < (1 - q)$ is also not possible. First, a media organization could only choose $\mu < (1 - q)$ if another organization also made the same choice (otherwise it would choose to shift inwards). Supposing a pair of media organizations chose $\mu_1 = \mu_2 = \chi < (1 - q)$, then there must be another organization choosing $\mu_3 = 3\chi$ – otherwise one of the fringe media organizations would deviate to $\mu = \chi + \epsilon$, for some $\epsilon > 0$.

The media organization $m = 3$ must choose this ideology to prevent $m = 1, 2$ from deviating. In turn, this requires that $\mu_4 = 5\chi$ – to prevent $m = 3$ from deviating. This continues across the spectrum of media organizations; $\mu_m = (2m - 3)\chi$ for $m \in \{3, \dots, M - 2\}$.

Note that for $\mu_{fr}(q, M) < (1 - q)$ to be sustainable as an equilibrium, there must also be a pair of media organizations choosing $\mu = 1 - \chi$ (an equally extreme ideology at the other end of the spectrum) – otherwise a fringe media organization could profitably deviate to $\mu = 1 - \chi + \epsilon$, for some $\epsilon > 0$. By identical logic to above, we require $\mu_{M-2} = 1 - 3\chi$.

Immediately above we found that $\mu_{M-2} = 1 - 3\chi$, but in the paragraph above found that, $\mu_{M-2} = (2M - 7)\chi$. For these to equate, we must have $\chi = \frac{1}{2M-4}$. But this is a contradiction – since we assumed that $\chi < \frac{1}{2M-4}$.²⁶

It now remains to note that $\mu_1^* = 1 - q$, $\mu_y^* = \frac{1-2q}{M-1}$ for $y \in \{2, \dots, M - 1\}$, $\mu_{M-1}^* = q$ is an equilibrium. So the fringe media ideology is $\mu_{fr}(q, M) = (1 - q)$, and such an equilibrium exists. \square

²⁶We did this by setting $\chi < (1 - q) \leq \frac{1}{2M-4}$ in the third paragraph.